Multi-Species Patlak-Keller-Segel System

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ABSTRACT. We study the regularity and large-time behavior of a crowd of species driven by chemo-tactic interactions. What distinguishes the different species is the way they interact with the rest of the crowd: the collective motion is driven by different chemical reactions which end up in a *coupled system* of parabolic Patlak-Keller-Segel equations. We show that the densities of the different species diffuse to zero provided the chemical interactions between the different species satisfy a certain sub-critical condition; the latter is intimately related to a log-Hardy-Littlewood-Sobolev inequality for systems due to Shafrir & Wolansky. Thus, for example, when two species interact, one of which has mass less than 4π, then the 2-system stays smooth for all time independent of the total mass of the system, in sharp contrast with the well-known breakdown of one species with initial mass $> 8\pi$.

CONTENTS

1. INTRODUCTION

In this paper, we consider the multi-species parabolic-elliptic Patlak-Keller-Segel (PKS) system which models chemotaxis phenomena involving multiple bacteria species

$$\begin{cases} \partial_t n_\alpha + \nabla \cdot (\nabla c_\alpha n_\alpha) = \Delta n_\alpha, & \alpha \in \mathcal{I}, \\ -\Delta c_\alpha = \sum_{\beta \in \mathcal{I}} b_{\alpha\beta} n_\beta, & \\ n_\alpha(x, t=0) = n_{\alpha 0}(x), & x \in \mathbb{R}^2. \end{cases} \tag{1.1}$$

Here, n_α, c_α denote the bacteria and the chemical densities, respectively. The parameters $\alpha, \beta \in \mathcal{I}$ indicate different species of bacteria/chemicals. The total number of species, which is denoted $|\mathcal{I}|$ throughout the paper, is assumed to be finite. The first equation in the system (1.1) describes the time evolution of the bacteria density n_α subject to chemical density distribution c_α and diffusion. The second equation governs the evolution of the chemical density c_α, which is determined by the collective effect of different species of bacteria n_β. The *chemical generation coefficients* $b_{\alpha\beta}$ represent the relative impact of the bacteria distribution n_β on the generation of the chemical c_α.

We comment that system (1.1) covers the more general setup, in which each species has its own sensitivity to the chemo-attractant, quantified by the positive constant parameters $\{\chi_\alpha\}$,

$$\begin{cases} \partial_t n_\alpha + \chi_\alpha \nabla \cdot (\nabla c_\alpha n_\alpha) = \Delta n_\alpha, & \alpha \in \mathcal{I}, \\ -\Delta c_\alpha = \sum_{\beta \in \mathcal{I}} b_{\alpha\beta} n_\beta, & \\ n_\alpha(x, t=0) = n_{\alpha 0}(x), & x \in \mathbb{R}^2. \end{cases} \tag{1.1$'$}$$

Indeed, if we let $\eta_\alpha > 0$ be scaling parameters at our disposal, we set $n'_\alpha := \eta_\alpha n_\alpha$ and $c'_\alpha := \chi_\alpha c_\alpha$. Then, (1.1)$'$ is reduced to (1.1) for the "tagged" variables, (n'_α, c'_α), with rescaled generation array, $b'_{\alpha\beta} = \chi_\alpha b_{\alpha\beta} \eta_\beta^{-1}$. In particular, choosing $\eta_\beta = 1/\chi_\beta$ shows that if $\mathbf{B} = \{b_{\alpha\beta}\}$ is symmetrical, then so is $\mathbf{B}'$.

In the last few years, social interaction within biofilms—a special form of bacteria colonies—has aroused increasing interest among the biology and biophysics community [12]. In a biofilm, billions of bacteria of different species live together and create hard-to-remove infections. Different cells in the biofilm specialize in various tasks, including acquiring food, defending the colony, and preserving genetic information. Chemical signals and ion signals are generated to communicate information within these bacteria colonies. The multi-species PKS model (1.1) serves as an attempt to understand the biofilm. Moreover, in the Chemotaxis experiment, the bacteria involved have large genetic variation. For example, E. coli only share 30% of their genes. Equation (1.1) serves as a more accurate model than single-species dynamics, taking into account the possible genetic variation appearing in the experiments.

We recall the large literature on the single species PKS model (1.1) ($|\mathcal{I}| = 1$), referring the interested reader to the review [18] and the following works: [3–6], [10, 11], [19], [17], [24], [23], [26], [20]. We summarize the essential results here. The preserved total mass of the solution $M := |n(t)|_{L^1} = |n_0|_{L^1}$ determines the long time behavior. If the intitial data n_0 has subcritical mass $M < 8\pi$ and finite second moment, the unique global smooth solutions exist for all time, [5], [7], [13]. If M is strictly greater than 8π and the second moment is finite, solution blows up in finite time, [19], [22], [5]. If $M = 8\pi$, the solution aggregates to a Dirac mass as time tends to infinity [4].

The multi-species PKS equation (1.1) has attracted increasing interest in the last decade. Its study originates in Wolansky's work [27]. Since then, much research has been carried out in the specific case of two interacting species [9], [2], [21], [1], [15], [14]. Even in the two-species case, the PKS systems (1.1) behave differently from the single-species ones. Consider the PKS equation (1.1) subject to symmetrical chemical generation coefficients

$$\mathbf{B} := \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{1.2}$$

which models two species with *cross-attractions*. We will prove that if one species has mass strictly less than 4π, the solutions to (1.1) exist globally regardless of the mass of the other species. However, if some critical mass constraint is violated, the solutions undergo finite time blow-up. On the other hand, for some special non-symmetrical chemical generation matrices, for example, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$, the solutions $\mathbf{n} := \{n_\alpha\}_{\alpha\in\mathcal{I}}$ to (1.1) decay to zero unconditionally.

In this paper, we quantify a global well-posedness condition for the multi-species PKS model (1.1) subject to symmetrical chemical generation coefficients, and we characterize its longtime behavior (for both—symmetrical and non-symmetrical cases), along the lines of our results announced in [16].

Before stating the main theorems, we list the basic assumptions and terminologies. The following initial conditions are always assumed:

$$\sum_{\alpha\in\mathcal{I}} n_{\alpha 0}(1 + |x|^2) \in L^1(\mathbb{R}^2); \quad n_{\alpha 0}\log n_{\alpha 0} \in L^1(\mathbb{R}^2), \ \forall\, \alpha \in \mathcal{I}. \tag{1.3}$$

We store the chemical generation coefficients $b_{\alpha\beta}$ and the masses

$$M_\alpha = |n_\alpha(\cdot, t)|_1 \equiv |n_{\alpha 0}|_1$$

in compact matrix/vector form:

$$\begin{aligned} \mathbf{B} &:= \{b_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{I}}, & \mathbf{B}_+ &:= \{(b_{\alpha\beta})_+\}_{\alpha,\beta\in\mathcal{I}}, \\ \mathbf{M} &:= \{M_\alpha\}_{\alpha\in\mathcal{I}}, & M_\alpha &= |n_{\alpha 0}|_1, \end{aligned}$$

where $(\cdot)_+$ denotes the positive part of the function. We introduce the function $Q_{\mathbf{B},\mathbf{M}}$ acting on subsets $\mathcal{J}$ of the index set $\mathcal{I}$,

$$Q_{\mathbf{B},\mathbf{M}}[\mathcal{J}] = \frac{\sum\limits_{\alpha,\beta\in\mathcal{J}} b_{\alpha\beta}M_\alpha M_\beta}{\sum\limits_{\alpha\in\mathcal{J}} M_\alpha}, \quad \mathcal{J} \subset \mathcal{I}. \tag{1.4}$$

In particular, if $\mathcal{J} = \mathcal{I}$, then $Q_{\mathbf{B},\mathbf{M}}[\mathcal{J}]$ has a simple matrix representation: namely, $Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}] = \langle \mathbf{BM}, \mathbf{M}\rangle / |\mathbf{M}|_1$, where $\langle \cdot, \cdot \rangle$, $|\cdot|_1$ denote the Euclidean inner product and the ℓ^1-vector norm.

We first studied the multi-species PKS system (1.1) subject to *symmetrical* arrays

$$b_{\alpha\beta} = b_{\beta\alpha}, \quad \forall\, \alpha, \beta \in \mathcal{I}. \tag{1.5}$$

As in the single species case, there exists natural dissipated free energy for the system (1.1):

$$\begin{aligned} E[\mathbf{n}] = \sum_{\alpha\in\mathcal{I}} \int n_\alpha \log n_\alpha \,\mathrm{d}x \\ + \sum_{\alpha,\beta\in\mathcal{I}} \frac{b_{\alpha\beta}}{4\pi} \iint n_\alpha(x) \log|x-y| n_\beta(y)\,\mathrm{d}x\,\mathrm{d}y, \\ \mathbf{n} := (n_\alpha)_{\alpha\in\mathcal{I}}. \end{aligned} \tag{1.6}$$

The proof of the dissipation of (1.6) is postponed to the next section. We solve the equation (1.1) in the distribution sense with a free energy dissipation constraint.

Definition 1.1 (Free energy solutions). For any distributional solutions $\mathbf{n}$ to the equation (1.1) subject to initial data $\mathbf{n}_0$, these are the free energy solutions to (1.1) if the following free energy dissipation inequality holds on some maximal time interval $[0, T_\star)$:

$$E[\mathbf{n}(t)] + \sum_{\alpha\in\mathcal{I}} \int_0^t \int_{\mathbb{R}^2} n_\alpha |\nabla \log n_\alpha - \nabla c_\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant E[\mathbf{n}_0], \tag{1.7}$$

for all $t \in [0, T_\star)$. If the equality in (1.7) is satisfied, we call it free energy dissipation equality.

The existence and blow-up theorems of (1.1) are stated as follows.

Theorem 1.2 (Global existence: subcritical mass). *Consider the equation (1.1) subject to initial conditions (1.3). If the symmetrical chemical generation matrix* $\mathbf{B}$ ($\mathbf{B}_+ \neq \mathbf{0}$) *and the mass vector* $\mathbf{M}$ *satisfy the subcritical mass constraint*

$$Q_{\mathbf{B}_+,\mathbf{M}}[\mathcal{I}] < 8\pi, \tag{1.8a}$$

$$Q_{\mathbf{B}_+,\mathbf{M}}[\mathcal{J}] < Q_{\mathbf{B}_+,\mathbf{M}}[\mathcal{I}] \quad \forall\, \varnothing \neq \mathcal{J} \subsetneqq \mathcal{I}, \tag{1.8b}$$

then the free energy solutions to (1.1) *exist for all finite time.*

The multi-species mass condition (1.8) recovers the threshold for global regularity of a single species (after rescaling), $\chi M < 8\pi$, which is known to be sharp [5, 7, 13, 19, 22]. It also provides a sharp characterization for global regularity of two-species dynamics.

Following are three prototypical examples.

Example 1.3 (Competition of two species). We consider the 2-species dynamics (1.2) with general sensitivity coefficients $\chi_1, \chi_2 > 0$,

$$\begin{gathered}\partial_t n_1 + \chi_1 \nabla \cdot (n_1 \nabla c_1) = \Delta n_1,\\ \partial_t n_2 + \chi_2 \nabla \cdot (n_2 \nabla c_2) = \Delta n_2,\\ \begin{cases} -\Delta c_1 = n_2,\\ -\Delta c_2 = n_1.\end{cases}\end{gathered}$$

Note that Theorem 1.2 applies to the rescaled variables $n'_\alpha = n_\alpha/\chi_\alpha$ with rescaled masses $M'_\alpha = M_\alpha/\chi_\alpha$ and the corresponding rescaled concentrations $c'_1 := \chi_1 c$ and $c'_2 := \chi_2 c$, coupled through the chemical generation array

$$\mathbf{B} = \begin{bmatrix} 0 & \chi_1\chi_2\\ \chi_1\chi_2 & 0\end{bmatrix}.$$

The sub-critical condition (1.8a) now reads $((\chi_2 M_1)^{-1} + (\chi_1 M_2)^{-1})^{-1} < 4\pi$, while (1.8b) is void since $Q_{\mathbf{B},\mathbf{M}'}[\mathcal{J}] = 0$ for $\mathcal{J} = \{1\}, \{2\}$. In particular, if the mass of one species—*either* $\chi_2 M_1$ *or* $\chi_1 M_2$—is strictly less than 4π, then (1.8) holds: global regularity follows *independently* of the mass of the other species.

Example 1.4 (Competition of three- and many-species). Consider the 3-species dynamics (1.2) with positive sensitivity coefficients $\chi_1 = \chi_3 := \chi$ and χ_2,

$$\begin{gathered}\partial_t n_\alpha + \chi_\alpha \nabla \cdot (n_\alpha \nabla c_\alpha) = \Delta n_\alpha, \qquad \alpha \in \{1, 2, 3\}\\ -\Delta \begin{bmatrix} c_1\\ c_2\\ c_3\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix} n_1\\ n_2\\ n_3\end{bmatrix}.\end{gathered}$$

Theorem 1.2 applies to the rescaled variables $n'_\alpha = n_\alpha/\chi_\alpha$ with rescaled masses $M'_\alpha = M_\alpha/\chi_\alpha$ and the corresponding rescaled chemical generation array

$$\mathbf{B} = \begin{bmatrix} 0 & \chi_1\chi_2 & 0\\ \chi_1\chi_2 & 0 & \chi_2\chi_3\\ 0 & \chi_2\chi_3 & 0\end{bmatrix}.$$

The sub-critical condition (1.8b) with $\mathcal{J} = \{1, 2\} \subset \{1, 2, 3\}$ requires

$$2\frac{M_1 M_2}{M_1/\chi_1 + M_2/\chi_2} < 2\frac{M_1 M_2 + M_2 M_3}{M_1/\chi_1 + M_2/\chi_2 + M_3/\chi_3},$$

which is satisfied for all the M_α (recalling that $\chi_3 = \chi_1$). Similarly, the sub-critical condition (1.8b) with $\mathcal{J} = \{2,3\} \subset \{1,2,3\}$ requires that

$$2\frac{M_2M_3}{M_2/\chi_2 + M_3/\chi_3} < 2\frac{M_1M_2 + M_2M_3}{M_1/\chi_1 + M_2/\chi_2 + M_3/\chi_3}$$

hold for all the M_α; finally, (1.8b) with $\mathcal{J} = \{1,3\}$ is void, and hence it remains to verify that (1.8a) holds:

$$2\frac{M_1M_2 + M_2M_3}{M_1/\chi_1 + M_2/\chi_2 + M_3/\chi_3} < 8\pi.$$

This inequality is satisfied if

$$\frac{1}{1/\chi_2M_1 + 1/\chi_1M_2} + \frac{1}{1/\chi_3M_2 + 1/\chi_2M_3} < 4\pi$$

For example, if $\chi M_2 < 2\pi$, then (1.8) holds, and global regularity follows *independently* of the mass of the pother species, M_1 and M_3.

This can be extended to a general many-species array

$$\begin{bmatrix} 0 & 1 & 0 & \dots & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & 1 \\ 0 & \dots & \ddots & 1 & 0 \end{bmatrix}.$$

Example 1.5 (Cooperation of two species). Consider now the 2-species dynamics [8, 14]

$$\begin{aligned} \partial_t n_1 + \chi_1 \nabla \cdot (n_1 \nabla c) &= \Delta n_1, \\ \partial_t n_2 + \chi_2 \nabla \cdot (n_2 \nabla c) &= \Delta n_2, \\ \Delta c + n_1 + n_2 - c &= 0. \end{aligned}$$

Theorem 1.2 applies to the rescaled variables $n'_\alpha = n_\alpha/\chi_\alpha$ with rescaled masses $M'_\alpha = M_\alpha/\chi_\alpha$ and the corresponding rescaled chemical generation array

$$\mathbf{B} = \begin{bmatrix} \chi_1^2 & \chi_1\chi_2 \\ \chi_1\chi_2 & \chi_2^2 \end{bmatrix}.$$

The sub-critical condition (1.8) now reads

$$\max\{\chi_1^2 M_1', \chi_2^2 M_2'\} < \frac{(\chi_1 M_1' + \chi_2 M_2')^2}{M_1' + M_2'} < 8\pi,$$

or, after scaling back,

$$\max\{\chi_1 M_1, \chi_2 M_2\} < \frac{(M_1 + M_2)^2}{M_1/\chi_1 + M_2/\chi_2} < 8\pi. \tag{1.9}$$

The inequality on the right of (1.9) coincides with the first part of characterization for global existence in [14, Theorem 1]. The inequality on the left of (1.9) holds whenever $\frac{1}{2} < \chi_1/\chi_2 < 2$ (independent of the M_i). Observe that (1.9) implies—and is therefore more restrictive than—the second part of the general characterization for global existence in [14, Theorem 1] which requires $\max\{\chi_1 M_1, \chi_2 M_2\} < 8\pi$.

While the last two examples show that the sub-critical mass condition (1.8b) may or may not be sharp for general $|\mathcal{I}| \geqslant 2$ species, the necessity of the upper-bound in (1.8a) is stated in the following.

Theorem 1.6 (Blow-up: supercritical mass). *Consider the equations* (1.1) *subject to smooth initial data $n_\alpha \in H^s$, $\forall\, \alpha \in \mathcal{I}$, $s \geqslant 2$, with finite second moment, and governed by a symmetrical chemical generation matrix* (1.5). *If $Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}] > 8\pi$, then the solution blows up at a finite time.*

Remark 1.7. Theorem 1.6 tells us that the bound $Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}] \leqslant 8\pi$ is necessary for existence of global-in-time free energy solution. A sufficient condition for this (strict) bound to hold is given by (consult Proposition 3.5 below)

$$\rho(\mathbf{B}_+) \max_\alpha M_\alpha < 8\pi, \quad \rho(X)_{|X \in \mathrm{Symm}_{\mathcal{I}\times\mathcal{I}}} := \max_\alpha \lambda_\alpha(X). \tag{1.10}$$

Thus, (1.10) implies that the first inequality (1.8a) is satisfied. As an example, we revisit the two-species example (1.2) (with $\chi_1 = \chi_2 = 1$). In this case, $Q_{\mathbf{B},\mathbf{M}}[\mathcal{J}] = 0$ for $\mathcal{J} \subsetneqq \mathcal{I}$, so the second inequalities in (1.8b) are void: it is only the first part, (1.8a), that needs to be verified. Here, $\rho(\mathbf{B}_+) = 1$ and the sufficient condition (1.10) amounts to $\max_{\alpha\in\{1,2\}} M_\alpha < 8\pi$, which suffices (yet stronger than the sharp $(M_1^{-1} + M_2^{-1})^{-1} < 4\pi$ encountered before) for (1.8a), and hence the global existence of (1.2).

To formulate the smoothness and uniqueness theorems, we need further physical restriction on the free energy solutions. First, the physical solutions to equation (1.1) should satisfy the conservation of mass:

$$|n_\alpha(t)|_1 \equiv |n_\alpha(0)|_1 = M_\alpha, \quad \forall\, \alpha \in \mathcal{I}, \ \forall\, t \in [0, T_\star). \tag{1.11a}$$

Moreover, by formal computation, which is postponed to the next section, we have that the total second moment of the physically relevant solutions should grow linearly:

$$\text{(1.11b)}\quad V[\mathbf{n}] := \sum_{\alpha\in\mathcal{I}} V_\alpha(t) = \sum_{\alpha\in\mathcal{I}} \int n_\alpha(x,t)|x|^2\,\mathrm{d}x$$
$$= \Big(\sum_\alpha 4M_\alpha\Big)\Big(1 - \frac{Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}]}{8\pi}\Big)t + \sum_{\alpha\in\mathcal{I}} V_\alpha(0), \qquad \forall\, t \in [0, T_\star).$$

Finally, as it is well known the boundedness of the entropy $S[n_\alpha] := \int_n^\alpha \log n_\alpha$ is closely related to existence of smooth solutions, we consider free energy solutions subject to bounded entropy and free energy dissipation:

$$\text{(1.11c)}\quad \mathcal{A}_t[\mathbf{n}] := \sup_{s\in[0,t]} \Big\{ \sum_{\alpha\in\mathcal{I}} \int n_\alpha(x,s)\log^+ n_\alpha(x,s)\,\mathrm{d}x \Big\}$$
$$+ \sum_{\alpha\in\mathcal{I}} \int_0^t \int n_\alpha(x,s)|\nabla \log n_\alpha(x,s) - \nabla c_\alpha(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s < \infty,$$

for all $t < T_\star$, where $T_\star$ denotes the maximal existing time and $\log^+$ denotes the positive part of the function log. A similar quantity is defined in the paper [13]. We say that a free energy solution is *physically relevant* if it satisfies physical constraints (1.11a), (1.11b), and (1.11c). Now we state the theorems concerning the smoothness, uniqueness, and long-time behavior of the physically relevant free energy solutions.

Theorem 1.8 (Smoothnness of the free energy solutions). *Consider the equations* (1.1) *subject to initial condition* (1.3) *and symmetrical chemical generation matrices* $\mathbf{B}$. *The physically relevant free energy solutions* $(n_\alpha)_{\alpha\in\mathcal{I}}$ *are smooth, that is,* $n_\alpha \in C^\infty((0,T_\star)\times\mathbb{R}^2)$, $\forall\,\alpha\in\mathcal{I}$, *where* $T_\star$ *is the maximal existence time. Moreover, the equality holds in* (1.7).

Theorem 1.9 (Uniqueness of the free energy solutions). *Consider the equation* (1.1) *subject to initial condition* (1.3) *and symmetrical chemical generation matrix* $\mathbf{B}$. *There exists at most one physically relevant free energy solution.*

Theorem 1.10 (Longtime behavior of the free energy solutions). *Consider the solutions to* (1.1) *subject to initial condition* $n_\alpha \in H^s$, $\forall\,\alpha\in\mathcal{I}$, $s \geqslant 2$, *and symmetrical chemical generation matrices* (1.5). *There exists a constant* C, *which only depends on the initial data, such that the following estimate is satisfied:*

$$\sum_{\alpha\in\mathcal{I}} |n_\alpha(t)|_2^2 \leqslant \frac{C}{1+t}, \qquad \forall\, t\in[0,\infty).$$

If the chemical generation matrix $\mathbf{B}$ is non-symmetrical, the free energy (1.6) defined above is no longer dissipated. As a result, we cannot use the machinery developed in [5] to prove a global well-posedness theorem. However, we can still prove the global existence and uniform-in-time boundedness results for the multi-species PKS systems (1.1) subject to a special class of chemical generation matrices which we call *essentially dissipative matrices*. The definition is as follows.

Definition 1.11. Define the sequences of subsets $\mathcal{I}^{(0)} \subset \mathcal{I}^{(1)} \subset \cdots \subset \mathcal{I}^{(|\mathcal{I}|)}$ of $\mathcal{I}$ as follows:

$$\begin{aligned}\mathcal{I}^{(0)} &:= \{\alpha \in \mathcal{I} \mid b_{\alpha\beta} \leqslant 0,\ \forall\, \beta \in \mathcal{I}\};\\ \mathcal{I}^{(k)} &:= \{\alpha \in \mathcal{I} \mid b_{\alpha\beta} \leqslant 0,\ \forall\, \beta \in \mathcal{I} \setminus \mathcal{I}^{(k-1)}\}, \quad k \in \{1, 2, \ldots, |\mathcal{I}|\}.\end{aligned}$$

If $\mathcal{I}^{(|\mathcal{I}|)} = \mathcal{I}$, we called the matrix $\mathbf{B}$ essentially dissipative.

Remark 1.12. The simplest essentially dissipative matrices $\mathbf{B}$ are

$$\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2\\ -1 & 0 & 3\\ -2 & -4 & 0\end{bmatrix}.$$

Essential dissipative matrices naturally arise when there are chasing-escaping phenomena in the multi-species PKS system (1.1). For example, the system (1.1) subject to chemical generation relation $b_{12} = -b_{21} = 1$, $b_{11} = b_{22} = 0$ describes the situation where bacteria of species 1 are escaping from bacteria of species 2, whereas bacteria of species 2 are chasing bacteria of species 1.

The theorem corresponding to the multi-species PKS model (1.1) subject to essentially dissipative $\mathbf{B}$ is as follows.

Theorem 1.13 (Non-symmetrical interactions). *Consider the multi-species* PKS *system* (1.1) *subject to initial condition* $(n_\alpha)_0 \in H^s$, $\forall\, \alpha \in \mathcal{I}$, $s \geqslant 2$. *Assume the chemical generation matrix* $\mathbf{B}$ *is essentially dissipative. Then, there exists a uniformly bounded* H^s *solution to the equation* (1.1) *for all time; that is, there exists a constant* $C_{H^s} = C_{H^s}(\{n_{\alpha 0}\}_{\alpha\in\mathcal{I}})$ *such that* $\sum_{\alpha\in\mathcal{I}} |n_\alpha(t)|_{H^s} \leqslant C_{H^s} < \infty$, *for all* $t \in [0,\infty)$. *Furthermore, there exists a constant* C, *which depends only on the initial data and* $\mathbf{B}$, *such that the following estimate is satisfied:*

$$\sum_{\alpha\in\mathcal{I}} |n_\alpha(t)|_2^2 \leqslant \frac{C}{1+t}, \qquad \forall\, t \geqslant 0.$$

The paper is organized as follows. In Section 2, we give preliminaries and the proof of Theorem 1.6. In Section 3, we prove the existence of global free energy solutions with subcritical mass. In Section 4, we prove the smoothness of the free energy solutions. In Section 5, we prove the uniqueness of the free energy solutions. In Section 6, we explore the long-time behavior of the free energy solutions. Finally, in the last section, we discuss the non-symmetrical case.

1.1. Notation. In the paper, we use the notation $A \lesssim B$ $(A, B \geqslant 0)$, if there exists a constant C such that $A \leqslant CB$. We will also use $\sum_\alpha$ to represent

$\sum_{\alpha\in\mathcal{I}}$ unless otherwise stated. Constants C_S, C_{HLS}, C_{lHLS}, C_{GNS}, and C_N are used to represent universal constant depending on various differential(integral) inequalities. The exact values might change from line to line. Given a vector $\mathbf{w}$ we let $|\mathbf{w}|_p$ denote its ℓ^p norm; given a vector function $\mathbf{w}(\cdot)$ we let $|\mathbf{w}(\cdot)|_X$ denote its norm in vector space X. In particular, $|\mathbf{w}(\cdot)|_p$ denote the usual L^p spaces, and the distinction between ℓ^p and L^p spaces is clear from the text.

2. PRELIMINARIES

Two quantities are crucial in the analysis of the multi-species PKS dynamics (1.1): the free energy $E[\mathbf{n}]$ (1.6) and the second moment $\sum_\alpha V_\alpha$ (1.11b). In this section, we calculate the time evolution of these two quantities formally, and give the proof of Theorem 1.6.

As in the single species case, the free energy $E[\mathbf{n}]$ (1.6) is formally dissipated under the equation (1.1).

Lemma 2.1. *Consider smooth solutions* $\mathbf{n}$ *to the equation* (1.1) *subject to initial data* $\mathbf{n}_0$ *and symmetrical* $\mathbf{B}$. *The free energy* $E[\mathbf{n}]$ (1.6) *is decreasing and it satisfies the free energy dissipation equality*

$$
\begin{aligned}
E[\mathbf{n}(t)] &= E[\mathbf{n}_0] - \sum_{\alpha\in\mathcal{I}} \int_0^t \int n_\alpha |\nabla \log n_\alpha - \nabla c_\alpha|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&=: E[\mathbf{n}_0] - \int_0^t \mathcal{D}[\mathbf{n}(s)] \, \mathrm{d}s.
\end{aligned} \tag{2.1}
$$

Proof. We apply the equation (1.1) and the symmetrical condition (1.5) to calculate the time evolution of the free energy $E[\mathbf{n}]$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E[\mathbf{n}] &= \sum_\alpha \int (n_\alpha)_t \log n_\alpha - \sum_\alpha \int \frac{c_\alpha (n_\alpha)_t}{2} \, \mathrm{d}x \\
&\quad - \sum_\alpha \int \frac{(c_\alpha)_t n_\alpha}{2} \, \mathrm{d}x \\
&= \sum_\alpha \int (n_\alpha)_t \log n_\alpha - \sum_\alpha \int \frac{c_\alpha (n_\alpha)_t}{2} \, \mathrm{d}x \\
&\quad + \sum_{\alpha,\beta} \frac{b_{\alpha\beta}}{4\pi} \int (n_\beta)_t(y) \log|x-y| n_\alpha(x) \, \mathrm{d}x \, \mathrm{d}y \\
&= \sum_\alpha \int (n_\alpha)_t \log n_\alpha - \sum_\alpha \int \frac{c_\alpha (n_\alpha)_t}{2} \, \mathrm{d}x \\
&\quad + \sum_{\alpha,\beta} \frac{b_{\alpha\beta}}{4\pi} \int (n_\alpha)_t(x) \log|x-y| n_\beta(y) \, \mathrm{d}x \, \mathrm{d}y \\
&= \sum_\alpha \int (n_\alpha)_t (\log n_\alpha - c_\alpha) \, \mathrm{d}x.
\end{aligned} \tag{2.2}
$$

Equation (1.1) can be rewritten as $\partial_t n_\alpha = \nabla \cdot (n_\alpha(\nabla \log n_\alpha - \nabla c_\alpha))$, and so applying integration by parts on the time evolution of $E[\mathbf{n}]$ (2.2) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} E[\mathbf{n}] = -\sum_\alpha \int n_\alpha |\nabla \log n_\alpha - \nabla c_\alpha|^2 \, \mathrm{d}x \leqslant 0.$$

Now, by integration in time, we obtain (2.1). □

Next, we give the time evolution of the second moment.

Lemma 2.2. *Consider the smooth solutions* $\mathbf{n}$ *to the equation* (1.1) *subject to smooth initial data* $\mathbf{n}_0 \in H^s$, $s \geqslant 2$, *and symmetrical chemical generation matrix* $\mathbf{B}$. *The time evolution of the total second moment* $\sum_{\alpha \in \mathcal{I}} V_\alpha$ (1.11b) *satisfies the equality*

$$\frac{\mathrm{d}}{\mathrm{d}t} V[\mathbf{n}] = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{\alpha \in \mathcal{I}} V_\alpha = \Big(\sum_{\alpha \in \mathcal{I}} 4M_\alpha \Big) \left(1 - \frac{Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}]}{8\pi} \right), \tag{2.3}$$

where $Q_{\mathbf{B},\mathbf{M}}$ *is defined in* (1.4).

Proof. Applying the equation (1.1), the definition of $Q_{\mathbf{B},\mathbf{M}}$ (1.4), and the symmetry condition (1.5), we calculate the time evolution of the total second moment as follows:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha V_\alpha &= \sum_\alpha 4M_\alpha + \sum_\alpha \int 2x \cdot (\nabla c_\alpha n_\alpha) \, \mathrm{d}x \\
&= \sum_\alpha 4M_\alpha - \sum_{\alpha,\beta} b_{\alpha\beta} \frac{1}{2\pi} \iint \frac{2x \cdot (x-y)}{|x-y|^2} n_\beta(y) n_\alpha(x) \, \mathrm{d}x \, \mathrm{d}y \\
&= \sum_\alpha 4M_\alpha - \sum_{\alpha,\beta} b_{\alpha\beta} \frac{1}{4\pi} \iint \frac{2(x-y) \cdot (x-y)}{|x-y|^2} n_\beta(y) n_\alpha(x) \, \mathrm{d}x \, \mathrm{d}y \\
&= \sum_\alpha 4M_\alpha - \sum_{\alpha,\beta} b_{\alpha\beta} \frac{M_\alpha M_\beta}{2\pi} \\
&= \Big(\sum_\alpha 4M_\alpha \Big) \left(1 - \frac{Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}]}{8\pi} \right).
\end{aligned}$$

This completes the proof of the lemma. □

Remark 2.3. Note that in the proofs of these two lemmas, the symmetry of the matrix $\mathbf{B}$ is always assumed. In the non-symmetrical case (i.e., $b_{\alpha\beta} \neq b_{\beta\alpha}$), neither of these lemmas can be applied. This is the main difficulty faced when applying the free energy machinery in the non-symmetrical case.

Proof of Theorem 1.6. Suppose that the solution $\mathbf{n}$ is smooth for all time. By the assumption $Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}] > 8\pi$, we have that the time evolution (2.3) is a strictly negative constant. As a result, the total second moment will decrease to zero at a finite time $T_\star$ while the L^1 norm of the solution $\sum_{\alpha\in\mathcal{I}} |n_\alpha|_1$ is preserved. At time $T_\star$, the smoothness assumption of the solution will be contradicted. Hence, the solution must lose H^s regularity before $T_\star$. ◻

3. GLOBAL EXISTENCE FOR SUBCRITICAL DATA

3.1.* A priori *estimate on entropy. In the case of a single species, the analysis of PKS equation proceeds by combining an *a priori* estimate of the free energy (1.7) together with a logarithmic Hardy-Littlewood-Sobolev inequality to recover a uniform-in-time *a priori* bound on the entropy, which in turn yields existence of a free energy solution for all time. In the present context of a *coupled* system of PKS equations, one seeks the corresponding log-Hardy-Littlewood-Sobolev inequality for systems which guarantees a finite lower bound of the multi-species functional $\Psi[\mathbf{n}]$, $\mathbf{n} := \{n_\alpha\}_{\alpha\in\mathcal{I}}$,

$$\Psi[\mathbf{n}] := \sum_{\alpha\in\mathcal{I}} \int_{\mathbb{R}^2} n_\alpha \log n_\alpha \,\mathrm{d}x + \frac{1}{4\pi} \sum_{\alpha,\beta\in\mathcal{I}} a_{\alpha\beta} \iint_{\mathbb{R}^2\times\mathbb{R}^2} n_\alpha(x) \log|x-y| n_\beta(y)\,\mathrm{d}x\,\mathrm{d}y,$$

over all n_α in the function space

$$\Gamma_{\mathbf{M}}(\mathbb{R}^2) = \Big\{(n_\alpha)_{\alpha\in\mathcal{I}} n_\alpha \geqslant 0 \mid \int_{\mathbb{R}^2} n_\alpha |\log n_\alpha|\,\mathrm{d}x < \infty,\ \int_{\mathbb{R}^2} n_\alpha\,\mathrm{d}x = M_\alpha, \int_{\mathbb{R}^2} n_\alpha \log(1+|x|^2)\,\mathrm{d}x < \infty,\ \forall\,\alpha\in\mathcal{I}\Big\}.$$

To this end, we follow [25]. For an arbitrary subset of our index set, $\mathcal{J}\subset\mathcal{I}$, one defines the quantity

$$\Lambda_{\mathcal{J}}(\mathbf{M}) := 8\pi \sum_{\alpha\in\mathcal{J}} M_\alpha - \sum_{\alpha,\beta\in\mathcal{J}} a_{\alpha\beta} M_\alpha M_\beta, \qquad \mathbf{M} := (M_\alpha)_{\alpha\in\mathcal{I}},\ |\mathcal{I}| < \infty.$$

Theorem 3.1 ([25, Theorem 4]). *Let* $\mathbf{A} = (a_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ *be a symmetrical matrix with positive entries* $a_{\alpha\beta} \geqslant 0$.

(a) *We note that*

$$\begin{cases} \Lambda_{\mathcal{I}}(\mathbf{M}) = 0, & \\ \Lambda_{\mathcal{J}}(\mathbf{M}) \geqslant 0, & \forall\,\varnothing \neq \mathcal{J}\subset\mathcal{I}, \\ \text{If } \Lambda_{\mathcal{J}}(\mathbf{M}) = 0 \text{ for some } \mathcal{J}, & \\ \ \text{then } a_{\alpha\alpha} + \Lambda_{\mathcal{J}\setminus\{\alpha\}}(\mathbf{M}) > 0, & \forall\,\alpha\in\mathcal{J}, \end{cases} \tag{3.1}$$

is a necessary and sufficient condition for the lower-bound of the PKS *functional* $\min_{\mathbf{n}\in\Gamma_{\mathbf{M}}(\mathbb{R}^2)}\Psi[\mathbf{n}]$.

(b) *Moreover, the functional* $\Psi[\mathbf{n}]$ *admits a minimizer over* $\Gamma_{\mathbf{M}}(\mathbb{R}^2)$ *if and only if* $\Lambda_{\mathcal{I}}(\mathbf{M}) = 0$ *and* $\Lambda_{\mathcal{J}}(\mathbf{M}) > 0$ *for any* $\emptyset \neq \mathcal{J} \subsetneq \mathcal{I}$. *In this case, there exists a constant,* $C = C_{\mathrm{lHLS}}$ *depending on* $\mathbf{M}$ *and* $\mathbf{B} = \{b_{\alpha\beta}\}$, *such that the following holds:*

$$\Psi[\mathbf{n}] \geqslant -C_{\mathrm{lHLS}}(\mathbf{M},\mathbf{B})\,. \tag{3.2}$$

Remark 3.2. As noted in [25, p. 414], if the condition $\Lambda_{\mathcal{J}} \geqslant 0$ is violated for some $\emptyset \neq \mathcal{J} \subsetneq \mathcal{I}$, then a scaling argument yields that the functional $\Psi[\mathbf{n}]$ on the sphere S^2 has no lower bound. One might be able to use this property to construct blow-up solutions *on the plane*, when the following strict monotonicity fails (recalling the functional $Q_{\mathbf{B}_+,\mathbf{M}}$ in (1.4)): $Q_{\mathbf{B}_+,\mathbf{M}}(\mathcal{J}) < Q_{\mathbf{B}_+,\mathbf{M}}(\mathcal{I})$ for all $\mathcal{J} \subsetneq \mathcal{I}$.

The above theorem yields the following result.

Proposition 3.3. *Consider the equation* (1.1) *subject to smooth initial data and chemical generation coefficient matrix* $\mathbf{B}$. *Further assume that* $\mathbf{B}_+$ *is not a zero matrix. Suppose that* (1.8) *holds, so* $Q_{\mathbf{B}_+,\mathbf{M}}[\mathcal{J}] < Q_{\mathbf{B}_+,\mathbf{M}}[\mathcal{I}] < 8\pi$, *and* $\emptyset \neq \mathcal{J} \subsetneq \mathcal{I}$. *Then, the total entropy* $\sum_\alpha \int n_\alpha \log n_\alpha \,\mathrm{d}x$ *is bounded for all finite time.*

Remark 3.4. We will not lose generality if we assume that $\mathbf{B}_+$ is not a zero matrix. If all the entries in $\mathbf{B}$ are negative, classical techniques are sufficient to analyze the system.

Proof. First, we rewrite the free energy dissipation relation (2.1) as follows:

$$\begin{aligned}
E[\mathbf{n}_0] \geqslant E[\mathbf{n}] \geqslant{}& \sum_{\alpha\in\mathcal{I}} \int n_\alpha \log n_\alpha \,\mathrm{d}x \\
&+ \sum_{\alpha,\beta\in\mathcal{I}} \frac{(b_{\alpha\beta})_+}{4\pi} \iint n_\alpha(x)\log|x-y| n_\beta(y)\,\mathrm{d}x\,\mathrm{d}y \\
&- \sum_{\alpha,\beta\in\mathcal{I}} \frac{(b_{\alpha\beta})_-}{4\pi} \iint_{|x-y|\geqslant 1} n_\alpha(x)\log|x-y| n_\beta(y)\,\mathrm{d}x\,\mathrm{d}y \\
={}& (1-\theta)\sum_{\alpha\in\mathcal{I}} \int n_\alpha \log n_\alpha \,\mathrm{d}x + \theta\Big(\sum_{\alpha\in\mathcal{I}} \int n_\alpha \log n_\alpha \,\mathrm{d}x \\
&+ \frac{1}{4\pi}\sum_{\alpha,\beta\in\mathcal{I}} \frac{(b_{\alpha\beta})_+}{\theta} \iint n_\alpha(x)\log|x-y| n_\beta(y)\,\mathrm{d}x\,\mathrm{d}y\Big) \\
&- \sum_{\alpha,\beta\in\mathcal{I}} \frac{(b_{\alpha\beta})_-}{4\pi}(M_\alpha V_\beta + M_\beta V_\alpha).
\end{aligned}$$

Define $a_{\alpha\beta} := (b_{\alpha\beta})_+/\theta \geqslant 0$, $0 < \theta < 1$.

In order to apply Theorem 3.1, we need to check the condition (3.1). By choosing θ properly, we make sure that the first condition $\Lambda_{\mathcal{I}}(\mathbf{M}) = 0$ in (3.1) is satisfied. Direct calculation yields that

$$\Lambda_{\mathcal{I}}(\mathbf{M}) = 0 \iff \theta = \frac{\sum_{\alpha,\beta\in\mathcal{I}} (b_{\alpha\beta})_+ M_\alpha M_\beta}{8\pi \sum_{\beta\in\mathcal{I}} M_\beta} = \frac{Q_{\mathbf{B}_+,\mathbf{M}}[\mathcal{I}]}{8\pi}.$$

Note that the assumption $Q_{\mathbf{B}_+,\mathbf{M}}(\mathcal{I}) < 8\pi$ guarantees that $\theta < 1$. Next, we check the remaining conditions in (3.1). Recalling the definition of θ and $Q_{\mathbf{B}_+,\mathbf{M}}[\mathcal{J}]$, the following condition guarantees the existence of the minimizer of Ψ in $\Gamma_{\mathbf{M}}(\mathbb{R}^2)$:

$$\begin{aligned}
&Q_{\mathbf{B}_+,\mathbf{M}}[\mathcal{I}] > Q_{\mathbf{B}_+,\mathbf{M}}[\mathcal{J}], \quad \forall\, \varnothing \neq \mathcal{J} \subsetneqq \mathcal{I},\\
&\iff \Lambda_{\mathcal{J}}(\mathbf{M}) = 8\pi \sum_{\beta\in\mathcal{J}} M_\beta - \frac{8\pi \sum_{\beta\in\mathcal{I}} M_\beta}{\sum_{\alpha,\beta\in\mathcal{I}} (b_{\alpha\beta})_+ M_\alpha M_\beta}\\
&\qquad\qquad \times \sum_{\alpha,\beta\in\mathcal{J}} (b_{\alpha\beta})_+ M_\alpha M_\beta > 0, \quad \forall\, \varnothing \neq \mathcal{J} \subsetneqq \mathcal{I},\\
&\iff \Lambda_{\mathcal{J}}(\mathbf{M}) > 0, \quad \forall\, \varnothing \neq \mathcal{J} \subsetneq \mathcal{I}.
\end{aligned}$$

Now combining Theorem 3.1, the boundedness of the second moment (2.3) and the fact that $0 < \theta < 1$ yields that

$$\begin{aligned}
E[\mathbf{n}_0] \geqslant E[\mathbf{n}] \geqslant (1-\theta) \sum_{\alpha\in\mathcal{I}} \int n_\alpha \log n_\alpha - \theta C_{\mathrm{lHLS}}\\
- \frac{1}{4\pi} \sum_{\alpha,\beta\in\mathcal{I}} (b_{\alpha\beta})_-(M_\alpha V_\beta + M_\beta V_\alpha),\\
\iff \sum_{\alpha\in\mathcal{I}} \int n_\alpha \log n_\alpha \,\mathrm{d}x \leqslant \frac{E[\mathbf{n}_0] + \theta C_{\mathrm{lHLS}} + \frac{1}{2\pi} \sum_{\alpha,\beta} (b_{\alpha\beta})_- M_\alpha V_\beta}{1-\theta} < \infty.
\end{aligned}$$

This completes the proof. □

The proof above shows that the log-HLS will not hold if $\mathrm{supp}(\mathbf{B}) \subsetneqq \mathcal{I}$, or else we can choose $\mathcal{J} = \mathrm{supp}(\mathbf{B}) \subsetneqq \mathcal{I}$ for which

$$\Lambda_{\mathcal{J}}(\mathbf{M}) = 8\pi \sum_{\beta\in\mathcal{J}} M_\beta - \frac{8\pi \sum_{\beta\in\mathcal{I}} M_\beta}{\sum_{\alpha,\beta\in\mathcal{I}} (b_{\alpha\beta})_+ M_\alpha M_\beta} \sum_{\alpha,\beta\in\mathcal{J}} (b_{\alpha\beta})_+ M_\alpha M_\beta < 0.$$

The precise characterization of $\mathbf{B}$ such that (1.8) holds remains open; consult our conjecture in Remark 7.2 below.

The precise characterization of $\mathbf{B}$ such that both conditions (1.8) hold remains open. We prove below the a sufficient condition, claimed in (1.10), for the upper-bound (1.8a) to hold.

Proposition 3.5. *Let* $\mathbf{A} = (a_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ *be a symmetrical matrix with positive entries* $a_{\alpha\beta} \geqslant 0$. *Then,* $Q_{\mathbf{A},M}[\mathcal{I}] < \rho(\mathbf{A}) \max_\alpha M_\alpha$.

To verify (3.1), we express $\mathbf{A}$ in terms of its spectral decomposition $\mathbf{A} = \sum_\alpha \lambda_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha^*$ where $\{(\lambda_\alpha, \mathbf{w}_\alpha)\}$ is the ortho-normal eigensystem of $\mathbf{A}$. We get

$$\langle \mathbf{AM}, \mathbf{M}\rangle = \sum_\alpha \lambda_\alpha |\langle \mathbf{M}, \mathbf{w}_\alpha\rangle|^2 \leqslant \max_\alpha \lambda_\alpha |\mathbf{M}|_2^2 \leqslant \max_\alpha \lambda_\alpha |\mathbf{M}|_1 \max_\alpha M_\alpha$$

and the result follows, $Q_{\mathbf{A},\mathbf{M}}[\mathcal{I}] \leqslant \rho(\mathbf{A}) \max_\alpha M_\alpha$.

3.2. Local existence and extension theorems. Before introducing the local existence theorems of the free energy solutions, we shall regularize the system (1.1) by appropriately truncating the singularity in the convolution kernel $\nabla K = \nabla(-\Delta)^{-1}$:

$$\begin{aligned} K^\varepsilon(z) &:= K^1\left(\frac{|z|}{\varepsilon}\right) - \frac{1}{2\pi}\log\varepsilon, \\ K^1(|z|) &:= -\frac{1}{2\pi}\log|z|, \quad |z| \geqslant 4. \\ K^1(|z|) &:= 0, \quad |z| \leqslant 1, \end{aligned}$$

We thus get the regularized multi-species PKS system

$$\begin{aligned} &\partial_t n_\alpha^\varepsilon + \nabla\cdot(\nabla c_\alpha^\varepsilon n_\alpha^\varepsilon) = \Delta n_\alpha^\varepsilon, \\ &c_\alpha^\varepsilon = K^\varepsilon * \Big(\sum_{\beta\in\mathcal{I}} b_{\alpha\beta} n_\beta\Big), \\ &n_\alpha^\varepsilon(t=0) = \min\{n_{\alpha 0}, \varepsilon^{-1}\}, \quad \forall\, \alpha\in\mathcal{I},\ x\in\mathbb{R}^2. \end{aligned}$$

Note that the masses of the solutions $M_\alpha = |n_\alpha|_1$ are preserved in time.

Since $|\nabla K^\varepsilon|_\infty$ is bounded for any fixed positive ε, applying Young's convolution inequality yields that the vector field ∇c_α is bounded in L^∞, that is, $\sum_\alpha |\nabla c_\alpha|_\infty \leqslant \sum_{\alpha,\beta} |\nabla K^\varepsilon|_\infty |b_{\alpha\beta}| M_\beta$. Now, standard convection-diffusion PDE theory can be applied to show that the regularized system (3.3) admits global solutions in $L^2((0,T];H^1) \cap C((0,T];L^2)$.

The following two propositions are the main local existence theorems.

Proposition 3.6 (Criterion for local existence). *Let* $(n_\alpha^\varepsilon)_{\alpha\in\mathcal{I}}$ *be the solutions to the regularized multi-species* PKS *system* (3.3) *on the time interval* $[0,T)$ *subject to initial constrain* (1.3). *If the total entropy* $\sum_\alpha S[n_\alpha^\varepsilon]$ *is bounded from above uniformly in* ε, *that is,*

$$\sum_{\alpha\in\mathcal{I}} S[n_\alpha^\varepsilon(t)] = \sum_{\alpha\in\mathcal{I}} \int n_\alpha^\varepsilon(x,t)\log n_\alpha^\varepsilon(x,t)\,\mathrm{d}x \leqslant C_{L\log L} < \infty, \quad \forall\, t\in[0,T],$$

then there exists a subsequence of $\{(n_\alpha^\varepsilon)_{\alpha\in\mathcal{I}}\}_{\varepsilon>0}$ *converging in the* $L_t^2L_x^2$ *strong topology to a non-negative free-energy solution to the multi-species* PKS *system* (1.1) *subject to initial data* $(n_\alpha)_0$ *on the time interval* $[0,T]$.

Proposition 3.7 (Blow-up criterion of free-energy solutions). *Consider the multi-species* PKS *system* (1.1) *subject to initial condition* (1.3). *There exists a maximal existence time* $T^* > 0$ *for the free-energy solution to the system* (1.1). *Moreover, if* $T^* < \infty$, *then there exists an* $\alpha \in \mathcal{I}$ *such that* $\lim_{t\to T^*}\int_{\mathbb{R}^2} n_\alpha(t)\log n_\alpha(t)\,\mathrm{d}x = \infty$.

Proof of Proposition 3.6. The proof is divided into three main steps.

Step 1: Here, we prove *a priori* estimates on mass distribution n^ε and chemical distribution c_α^ε to prepare for the following steps. For the reader's convenience, we summarize the uniform-in-ε estimates we obtained in this step:

$$\sum_\alpha |(1+|x|^2)n_\alpha^\varepsilon|_{L_t^\infty(0,T;L_x^1)} \leqslant C_V(\{(V_\alpha)_0\}_{\alpha\in\mathcal{I}},\mathbf{M}) < \infty; \tag{3.4a}$$

$$\sum_\alpha |n_\alpha^\varepsilon \log^\varepsilon n_\alpha^\varepsilon|_{L_t^\infty(0,T;L_x^1)} \leqslant C(C_{L\log L},C_V) < \infty; \tag{3.4b}$$

$$\sum_\alpha |\nabla\sqrt{n_\alpha}|^2_{L_t^2(0,T;L_x^2)} \leqslant C(C_{L\log L},C_V) < \infty; \tag{3.4c}$$

$$\sum_\alpha |\sqrt{n_\alpha}\,\nabla c_\alpha|^2_{L_t^2(0,T;L_x^2)} \leqslant C(C_{L\log L},C_V) < \infty. \tag{3.4d}$$

Before proving these estimates, recall the following Gagliardo-Nirenberg-Sobolev inequality, which is applied several times in the sequel:

$$|u|^2_{L^p} \leqslant C_{\mathrm{GNS}}|\nabla u|_{L^2}^{2-4/p}\,|u|_{L^2}^{4/p}, \quad \forall\, u\in H^1,\ \forall\, p\in[2,\infty).$$

We start by proving the second moment control of the solutions (3.4a). Much as in the calculation in the proof of Lemma 3.5, we have the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_\alpha\int n_\alpha|x|^2\,\mathrm{d}x\Big) \leqslant 4\sum_\alpha M_\alpha + \sum_{\alpha,\beta}(b_{\alpha\beta})_-\frac{M_\alpha M_\beta}{2\pi}, \tag{3.5}$$

from which the estimate (3.4a) follows directly.

To prove the L^1 control of $n_\alpha^\varepsilon\log n_\alpha^\varepsilon$ (3.4b), we recall the following lemma.

Lemma 3.8. *For any* g *where* $(1+|x|^2)g\in L_+^1(\mathbb{R}^2)$, *we have* $g\log^- g\in L^1(\mathbb{R}^2)$ *and*

$$\int_{\mathbb{R}^2} g\log^- g\,\mathrm{d}x \leqslant \frac{1}{2}\int_{\mathbb{R}^2} g(x)|x|^2\,\mathrm{d}x + \log(2\pi)\int_{\mathbb{R}^2} g(x)\,\mathrm{d}x + \frac{1}{e}. \tag{3.6}$$

Proof. The proof of the lemma can be found in the paper [5] and [4]. We refer the interested readers to these papers for further details. ∎

The estimate (3.6) yields that

$$\begin{aligned}\int |n_\alpha^\varepsilon \log n_\alpha^\varepsilon|\,\mathrm{d}x &\leqslant \int n_\alpha^\varepsilon(\log n_\alpha^\varepsilon + |x|^2)\,\mathrm{d}x + 2\log(2\pi)M_\alpha + \frac{2}{e}\\ &\leqslant C_{L\log L} + C_V + 2\log(2\pi)M_\alpha + \frac{2}{e}.\end{aligned}$$

As a result, we prove (3.4b).

Next, we show the bound of $|\nabla\sqrt{n_\alpha}|^2_{L^2_t(0,T;L^2_x)}$ (3.4c). This term naturally arises when we calculate the time evolution of the entropy $\sum_\alpha S[n_\alpha]$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\alpha\in\mathcal{I}} S[n_\alpha] = -4\sum_{\alpha\in\mathcal{I}}\int |\nabla\sqrt{n_\alpha}|^2\,\mathrm{d}x + \sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta}\int n_\alpha n_\beta\,\mathrm{d}x. \tag{3.7}$$

If we integrate (3.7), the quantity $\sum_\alpha |\nabla\sqrt{n_\alpha}|^2_{L^2_t(0,T;L^2_x)}$ will appear on the right-hand side. Therefore, we need to estimate the other terms in (3.7). Before going into the detailed estimates of the second term on the righthand side of (3.7), we recall that the total mass in the superlevel set can be estimated in terms of the entropy bound $C_{L\log L}$:

$$\begin{aligned}\sum_{\alpha\in\mathcal{I}}\int_{n_\alpha\geqslant K} n_\alpha\,\mathrm{d}x &\leqslant \frac{1}{\log(K)}\sum_{\alpha\in\mathcal{I}}\int |n_\alpha\log n_\alpha|\,\mathrm{d}x\\ &\leqslant \frac{C_{L\log L}}{\log(K)} =: \eta(K).\end{aligned} \tag{3.8}$$

If K is chosen large compared to the bound $C_{L\log L}$, the constant $\eta(K)$ will be small. It is classical to use this fact to control the nonlinearity in the PKS equation. Now, the second term on the righthand side of (3.7) can be estimated using Hölder's inequality, Gagliardo-Nirenberg-Sobolev inequality, and Young's inequality as follows:

$$\begin{aligned}&\sum_{\alpha,\beta} b_{\alpha\beta}\int n_\alpha n_\beta\,\mathrm{d}x\\ &\quad\leqslant \max_{\alpha,\beta}|b_{\alpha\beta}|\sum_\alpha |n_\alpha|_2\sum_\beta |n_\beta|_2\\ &\quad\leqslant \max_{\alpha,\beta}|b_{\alpha\beta}|\Big(\sum_\alpha |n_\alpha\mathbf{1}_{n_\alpha\geqslant K}|_2 + \sum_\alpha M_\alpha^{1/2}K^{1/2}\Big)^2\\ &\quad\leqslant 2\max_{\alpha,\beta}|b_{\alpha\beta}|\Big(\sum_\alpha |n_\alpha\mathbf{1}_{n_\alpha\geqslant K}|_1^{1/4}\,|n_\alpha|_3^{3/4}\Big)^2\\ &\qquad + 2\max_{\alpha,\beta}|b_{\alpha\beta}|\cdot|\mathcal{I}|K\sum_\alpha M_\alpha\\ &\quad\leqslant \eta(K)^{1/2}C_{\mathrm{GNS}}\max_{\alpha,\beta}|b_{\alpha\beta}|\Big(\sum_\alpha M_\alpha^{1/2}\Big)\Big(\sum_\alpha|\nabla\sqrt{n_\alpha}|_2^2\Big)\\ &\qquad + 2\max_{\alpha,\beta}|b_{\alpha\beta}|\cdot|\mathcal{I}|K\sum_\alpha M_\alpha.\end{aligned} \tag{3.9}$$

Combining (3.7) and (3.9), we have the following estimate on the time evolution of $\sum_\alpha S[n_\alpha]$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha S[n_\alpha] &\\ &\leqslant -\sum_\alpha\Big(4-\eta(K)^{1/2}C_{\mathrm{GNS}}\max_{\alpha,\beta}|b_{\alpha\beta}|\Big(\sum_\alpha M_\alpha^{1/2}\Big)\Big)|\nabla\sqrt{n_\alpha}|_2^2\\ &\quad + 2\max_{\alpha,\beta}|b_{\alpha\beta}|\cdot|\mathcal{I}|K\sum_\alpha M_\alpha.\end{aligned}$$

The coefficient

$$-(4-\eta(K)^{1/2}C_{\mathrm{GNS}}\max_{\alpha,\beta}|b_{\alpha\beta}|(\sum_\alpha M_\alpha^{1/2}))$$

is negative for K large enough. Therefore, for large enough K, we have the following estimate:

$$\begin{aligned}\sum_\alpha\int_0^T\int|\nabla\sqrt{n_\alpha}|^2\,\mathrm{d}x\,\mathrm{d}t &\\ &\leqslant \frac{S[\mathbf{n}(0)]-S[\mathbf{n}(T)]+2\max_{\alpha,\beta}|b_{\alpha\beta}|\cdot|\mathcal{I}|K\sum_\alpha M_\alpha T}{4-\eta(K)^{1/2}C_{\mathrm{GNS}}\max_{\alpha,\beta}|b_{\alpha\beta}|\Big(\sum_\alpha M_\alpha^{1/2}\Big)}<\infty.\end{aligned}$$

Since the entropy $S[\mathbf{n}(T)]$ is bounded, the righthand side is bounded. This completes the proof of (3.4c).

Finally, we prove the estimate (3.4d). The term $|\sqrt{n_\alpha^\varepsilon}\,\nabla c_\alpha^\varepsilon|_2^2$ naturally arises when we calculate the time evolution of $\sum_\alpha\int n_\alpha^\varepsilon c_\alpha^\varepsilon\,\mathrm{d}x$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha\int n_\alpha^\varepsilon c_\alpha^\varepsilon\,\mathrm{d}x=\sum_\alpha\int n_\alpha^\varepsilon\Delta c_\alpha^\varepsilon+\sum_\alpha\int n_\alpha^\varepsilon|\nabla c_\alpha^\varepsilon|^2\,\mathrm{d}x.$$

Integration in time yields that

$$\begin{aligned}\sum_\alpha\int_0^T\int n_\alpha^\varepsilon|\nabla c_\alpha^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t &= \frac{1}{2}\int n_\alpha^\varepsilon(T)c_\alpha^\varepsilon(T)-\frac{1}{2}\int n_\alpha^\varepsilon(0)c_\alpha^\varepsilon(0)\,\mathrm{d}x\\ &\quad -\sum_\alpha\int_0^T\int n_\alpha^\varepsilon\Delta c_\alpha^\varepsilon\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.10}$$

We first estimate the first term on the righthand side of (3.10). Applying the estimate of $|n_\alpha^\varepsilon\log n_\alpha^\varepsilon|_{L_t^\infty(0,T;L_x^1)}$ (3.4b), the relation $c_\alpha^\varepsilon=\sum_\beta b_{\alpha\beta}K^\varepsilon * n_\beta^\varepsilon$, and

Young's inequality $ab \leqslant e^{a-1} + b \ln b$, $\forall\, a, b \geqslant 1$, we deduce that

$$\begin{aligned}
|c_\alpha^\varepsilon(x)| &\leqslant \frac{1}{2\pi} \sum_{\beta\in\mathcal{I}} |b_{\alpha\beta}| \int_{|x-y|\leqslant 1} |K^\varepsilon(|x-y|) n_\beta(y)|\,\mathrm{d}y \\
&\quad + \frac{1}{2\pi} \sum_{\beta\in\mathcal{I}} |b_{\alpha\beta}| \int_{|x-y|\geqslant 1} |K^\varepsilon(|x-y|) n_\beta(y)|\,\mathrm{d}y \\
&\leqslant \sum_{\beta\in\mathcal{I}} |b_{\alpha\beta}| \int_{|x-y|\leqslant 1} \left((1+n_\beta(y)) \log(1+n_\beta(y)) + \frac{1}{|x-y|} \right) \mathrm{d}y \\
&\quad + \sum_{\beta\in\mathcal{I}} |b_{\alpha\beta}| \int (\log(1+|x|) + \log(1+|y|)) n_\beta(y)\,\mathrm{d}y \\
&\lesssim \sum_{\beta\in\mathcal{I}} |b_{\alpha\beta}| (C_{L\log L} + M_\beta + 1 + V_\beta + M_\beta \log(1+|x|)).
\end{aligned}$$

Combining it with the second moment control (3.4a), we have that $\int n_\alpha c_\alpha(t)$ is bounded independent of ε on time interval $[0, T]$:

$$\begin{aligned}
\text{(3.11)} \qquad & \int n_\alpha c_\alpha \,\mathrm{d}x \\
& \lesssim \sum_{\beta\in\mathcal{I}} |b_{\alpha\beta}| (C_{L\log L} + M_\beta + 1 + V_\beta) M_\alpha + \sum_{\beta\in\mathcal{I}} |b_{\alpha\beta}| M_\beta V_\alpha < \infty.
\end{aligned}$$

The last term on the righthand side of (3.10) can be estimated by using the $L^2([0,T]\times\mathbb{R}^2)$ estimate of $\nabla\sqrt{n_\alpha^\varepsilon}$ (3.4c) and the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha S[n_\alpha^\varepsilon(t)] = -4 \sum_\alpha \int |\nabla\sqrt{n_\alpha^\varepsilon}|^2\,\mathrm{d}x + \sum_{\alpha\in\mathcal{I}} \int n_\alpha^\varepsilon(-\Delta c_\alpha^\varepsilon)\,\mathrm{d}x.$$

Time integration of this relation yields that

$$\begin{aligned}
&\left| \sum_{\alpha\in\mathcal{I}} \int_0^T \int n_\alpha^\varepsilon(-\Delta c_\alpha^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad = \left| \sum_\alpha S[n_\alpha^\varepsilon(T)] - \sum_\alpha S[n_\alpha^\varepsilon(0)] + 4 \sum_\alpha \int_0^T \int |\nabla\sqrt{n_\alpha^\varepsilon}|^2\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad \leqslant C(C_{L\log L}) < \infty.
\end{aligned}$$

Combining this estimate, (3.10), and (3.11), we have completed the proof of (3.4d). In this way, we have obtained estimates on the two terms appearing in the dissipation of the free energy.

Step 2: Passing to the limit in $L_t^2(\delta, T; L^2)$ for any $\delta > 0$. Here, we would like to use the Aubin-Lions compactness lemma.

Lemma 3.9 (***Aubin-Lions lemma,*** [4]). *Take $T > 0$ and $1 < p < \infty$. Assume that $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence of functions in $L^p([0,T];H)$ where H is a Banach space. If $(f_n)_{n\in\mathbb{N}}$ is also bounded in $L^p([0,T];V)$ where V is compactly embedded in H and $(\partial f_n/\partial_t)_{n\in\mathbb{N}} \subset L^p([0,T];W)$ uniformly with respect to $n \in \mathbb{N}$ where H is imbedded in W, then $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $L^p([0,T];H)$.*

Our goal is to find the appropriate spaces V, H, W for $(n_\alpha^\varepsilon)_{\varepsilon>0}$. We subdivide the proof into steps; each step determines one space in the lemma. We will show that the following estimates are satisfied by the regularized solutions, with the constant $C_{L_t^2H_x^1}$ independent of the regularized parameter ε:

$$|n_\alpha^\varepsilon|_{L_t^2([\delta,T],L_x^2)} \leqslant C_{L_t^2H_x^1} < \infty,$$
$$|\nabla n_\alpha^\varepsilon|_{L_t^2([\delta,T],L_x^2)} \leqslant C_{L_t^2H_x^1} < \infty, \quad \forall\, \alpha \in \mathcal{I}.$$

We begin with the $H = L^2$- estimate of $\sum_\alpha |n_\alpha^\varepsilon|^2_{L_t^2([\delta,T];L_x^2)}$. Here, we prove that the solutions $n_\alpha^\varepsilon(t)$, $\forall\, \alpha \in \mathcal{I}$, are L^2 integrable in space for all $t \in [\delta, T]$. If the initial data $n_{\alpha 0}$ are L^2 integrable for all α, the solutions to the regularized equation (3.3) stay in L^2 for all time. This is the content of Lemma 3.10. However, the initial constraint (1.3) does not guarantee L^p boundedness, so we prove the hypercontractivity property of the equation (1.1), which yields that the solutions become L^2 integrable after an arbitrarily small amount of time $\delta > 0$. This is the content of Lemma 3.11.

Lemma 3.10. *Consider the regularized multi-species* PKS *system* (3.3) *subject to initial condition $n_{\alpha 0} \in L^p$, $\forall\, \alpha \in \mathcal{I}$, $\forall\, p \in [1,\infty)$. If the assumptions in Proposition 3.6 are satisfied, then the solutions to the system* (3.3) *are bounded in L^p for all $t \in [0, T]$.*

Proof. The $p = 1$ case is equivalent to the fact that the regularized equations preserve mass.

We do the L^p energy estimate formally; that is, we assume $-\Delta c_\alpha = \sum_\beta b_{\alpha\beta} n_\beta$, and refer the interested readers to the paper [5] for detailed justifications. During the calculation, we will use the following natural implication of the GNS inequality:

$$\begin{aligned}
\int (f-K)_+^{p+1}\,\mathrm{d}x &\leqslant C_{\mathrm{GNS}} \int (f-K)_+\,\mathrm{d}x \int |\nabla (f-K)_+^{p/2}|^2\,\mathrm{d}x \\
&\leqslant C_{\mathrm{GNS}} \frac{|f\log f|_1}{\log K} \int |\nabla (f-K)_+^{p/2}|^2\,\mathrm{d}x \\
&=: C_{\mathrm{GNS}}\,\eta(K) \int |\nabla (f-K)_+^{p/2}|^2\,\mathrm{d}x.
\end{aligned} \tag{3.12}$$

Note that if $|f \log f|_1$ is bounded, $\eta(K)$ is small if one chooses K large. Now, we estimate the time evolution of $\sum_\alpha |(n_\alpha - K)_+|_p^p$ with (3.12) as follows:

$$
\begin{aligned}
&\frac{1}{p}\sum_\alpha \frac{\mathrm{d}}{\mathrm{d}t}\int (n_\alpha - K)_+^p \,\mathrm{d}x \\
&\quad = -4\frac{p-1}{p^2}\sum_\alpha \int |\nabla (n_\alpha - K)_+^{p/2}|^2 \,\mathrm{d}x - \sum_\alpha \frac{1}{p}\int \nabla c_\alpha \cdot \nabla (n_\alpha - K)_+^p \,\mathrm{d}x \\
&\qquad - \sum_\alpha \int \Delta c_\alpha n_\alpha (n_\alpha - K)_+^{p-1} \,\mathrm{d}x \\
&\quad \leqslant -4\frac{p-1}{p^2}\sum_\alpha \int |\nabla (n_\alpha - K)_+^{p/2}|^2 \,\mathrm{d}x \\
&\qquad + \frac{p+1}{p}\sum_{\alpha,\beta} |b_{\alpha\beta}| \int (n_\alpha - K)_+^p (n_\beta - K)_+ \,\mathrm{d}x \\
&\qquad + \frac{p+1}{p} K \sum_{\alpha,\beta} |b_{\alpha\beta}| \int (n_\alpha - K)_+^p \,\mathrm{d}x \\
&\qquad + K \sum_{\alpha,\beta} |b_{\alpha\beta}| \int (n_\beta - K)_+ (n_\alpha - K)_+^{p-1} \,\mathrm{d}x \\
&\qquad + K^2 \sum_{\alpha,\beta} \int |b_{\alpha\beta}| (n_\alpha - K)_+^{p-1} \,\mathrm{d}x,
\end{aligned}
$$

and hence we find

$$
\begin{aligned}
&\frac{1}{p}\sum_\alpha \frac{\mathrm{d}}{\mathrm{d}t}\int (n_\alpha - K)_+^p \,\mathrm{d}x \\
&\quad \leqslant -4\frac{p-1}{p^2}\sum_\alpha \int |\nabla (n_\alpha - K)_+^{p/2}|^2 \,\mathrm{d}x \\
&\qquad + \max_\alpha \Big(\sum_\beta |b_{\alpha\beta}|\Big) C_{\mathrm{GNS}} \sum_\alpha |(n_\alpha - K)_+|_1 \, |\nabla (n_\alpha - K)_+^{p/2}|_2^2 \\
&\qquad + C_p(K, \mathbf{B}, \mathbf{M}) \, |(n_\alpha - K)_+|_p^p + C_p(K, \mathbf{B}, \mathbf{M}) \\
&\quad \leqslant \Big(-\frac{4(p-1)}{p^2} + \eta(K) \max_\alpha \Big(\sum_\beta |b_{\alpha\beta}|\Big) C_{\mathrm{GNS}} \Big) \\
&\qquad \times \sum_\alpha \int |\nabla (n_\alpha - K)_+^{p/2}|^2 \,\mathrm{d}x \\
&\qquad + C_p(K, \mathbf{B}, \mathbf{M}) \sum_\alpha |(n_\alpha - K)_+|_p^p + C_p(K, \mathbf{B}, \mathbf{M}).
\end{aligned}
$$

Because of the estimates (3.4b) and (3.8), the constant $\eta(K)$ can be made small enough such that the leading order term is negative, and the estimate can be further simplified as follows:

$$\begin{aligned}(3.13)\qquad &\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\alpha}|(n_\alpha-K)_+|_p^p\\ &\qquad\leqslant C_p(K,\mathbf{B},\mathbf{M})\sum_{\alpha}|(n_\alpha-K)_+|_p^p+C_p(K,\mathbf{B},\mathbf{M}).\end{aligned}$$

Now, we see that for any finite time interval $[0,T]$, the L^p norm is bounded uniformly independent of ε. ◻

Lemma 3.11. *Consider the regularized multi-species* PKS *system* (3.3) *subject to initial data* $\mathbf{n}_0$ *satisfying* (1.3). *If the assumptions in Proposition 3.6 are satisfied, then there exists a continuous function* $h_p\in C(\mathbb{R}_+)$ *such that for almost any* $t>0$, $|n(\cdot,t)|_p\leqslant h_p(t)$.

Proof. The proof is similar to the corresponding proof in [5] with some modifications. For the sake of completeness, we sketch the proof. First, we fix $t>0$ and $1<p<\infty$, and define

$$q(s):=1+(p-1)\frac{s}{t},\quad q\in[1,p],\ \text{for } s\in[0,t].$$

Next, we define the following quantities:

$$\begin{aligned}\mathbb{F}_\alpha(s)&=\left(\int_{\mathbb{R}^2}(n_\alpha(x,s)-K)_+^{q(s)}\,\mathrm{d}x\right)^{1/q(s)},\\ \mathbb{F}(s)&=\Big(\sum_{\alpha}\mathbb{F}_\alpha^{q(s)}(s)\Big)^{1/q(s)}.\end{aligned}$$

By taking the s derivative of the function $\mathbb{F}^{q(s)}(s)$, we obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}s}\sum_{\alpha}\int(n_\alpha(x,s)-K)_+^{q(s)}\,\mathrm{d}x=q(s)\mathbb{F}^{q(s)-1}\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{F}+\frac{\mathrm{d}q(s)/\mathrm{d}s}{q(s)}\mathbb{F}^{q(s)}\log\mathbb{F}^{q(s)}.$$

Combining this with the log-Sobolev inequality

$$\begin{aligned}&\int f^2\log\left(\frac{f^2}{\int f^2\,\mathrm{d}x}\right)\mathrm{d}x\\ &\qquad\leqslant 2\sigma\int|\nabla f|^2\,\mathrm{d}x-(2+\log(2\pi\sigma))\int f^2\,\mathrm{d}x,\quad\forall\,\sigma>0,\end{aligned}$$

and using the same argument to prove (3.13), we end up with the following estimate, inside which the notation $(\cdot)'$ is used to represent $\mathrm{d}/\mathrm{d}s$:

$$\begin{aligned}\mathbb{F}^{q-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{F} &= \frac{q'}{q^2}\sum_\alpha\int (n_\alpha-K)_+^q\log\frac{(n_\alpha-K)_+^q}{\mathbb{F}^q}\,\mathrm{d}x\\&\quad+\sum_\alpha\int(n_\alpha-K)_+^{q-1}\,\partial_s n_\alpha\,\mathrm{d}x\\&\leqslant\frac{q'}{q^2}\sum_\alpha\int (n_\alpha-K)_+^q\log\frac{(n_\alpha-K)_+^q}{\mathbb{F}_\alpha^q}\,\mathrm{d}x\\&\quad+\sum_\alpha\int(n_\alpha-K)_+^{q-1}\,\partial_s n_\alpha\,\mathrm{d}x\\&\leqslant\sum_\alpha\left(\frac{2\sigma q'}{q^2}-4\frac{q-1}{q^2}+C(\mathbf{B})\eta(K)\right)|\nabla(n_\alpha-K)_+^{q/2}|_2^2\\&\quad+\sum_\alpha\left((-2-\log(2\pi\sigma))\frac{q'}{q^2}+C(q,\mathbf{B},\mathbf{M},K)\right)\\&\quad\times\int(n_\alpha-K)_+^q\,\mathrm{d}x+C(q,\mathbf{B},\mathbf{M},K).\end{aligned}$$

Here, the constant $C(q,\mathbf{B},\mathbf{M},K)$ depends on the parameter q. However, since q is lying in a compact set $[0,p]$ on the time interval $[0,t]$, it can be chosen such that it only depends on the fixed parameter p. Now, by taking σ small enough, we end up with the following differential inequality:

$$\mathbb{F}^{q-1}\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{F}\leqslant\left((-2-\log(2\pi\sigma))\frac{q'}{q^2}+C(p,\mathbf{B},\mathbf{M},K)\right)\mathbb{F}^q+C(p,\mathbf{B},\mathbf{M},K).$$

Combining the fact that $\mathbb{F}(0)$ is finite and the coefficient

$$(-2-\log(2\pi\sigma))\frac{q'}{q^2}+C(p,\mathbf{B},\mathbf{M},K)$$

is time integrable on $[0,t]$, and applying standard ODE estimates, we obtain that $\mathbb{F}\leqslant h_p(t)$. This finishes the proof of the lemma. ◻

We now turn to the V-space estimates, where

$$V:=H^1\cap\left\{f\mid\int f|x|^2\,\mathrm{d}x<\infty\right\}:\sum_\alpha|\nabla n_\alpha^\varepsilon|^2_{L_t^2([\delta,T];L_x^2)}.$$

To get the $L_t^2([\delta,T];L_x^2)$ control of the $\nabla n_\alpha^\varepsilon$, we first calculate the time evolution of $\sum|n_\alpha^\varepsilon|_2^2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha\int|n_\alpha^\varepsilon|^2\,\mathrm{d}x=-2\sum_\alpha\int|\nabla n_\alpha^\varepsilon|^2\,\mathrm{d}x+2\sum_\alpha\int\nabla n_\alpha^\varepsilon\cdot\nabla c_\alpha^\varepsilon n_\alpha^\varepsilon\,\mathrm{d}x.$$

Integration in time yields that

$$\sum_\alpha |n^\varepsilon_\alpha(T)|_2^2 - \sum_\alpha |n^\varepsilon_\alpha(\delta)|_2^2 + \sum_\alpha |\nabla n^\varepsilon_\alpha|^2_{L^2_t([\delta,T];L^2_x)} \leqslant \sum_\alpha |n^\varepsilon_\alpha \nabla c^\varepsilon_\alpha|^2_{L^2_t([\delta,T];L^2_x)}. \tag{3.14}$$

We see that since $|n^\varepsilon_\alpha|_{L^\infty_t(\delta,T;L^2_x)}$ is bounded independent of ε, if the righthand side $\sum_\alpha |n^\varepsilon_\alpha \nabla c^\varepsilon_\alpha|_{L^2_t([\delta,T];L^2_x)}$ is bounded, $|\nabla n^\varepsilon_\alpha|_{L^2_t(\delta,T;L^2_x)}$ will be bounded independent of ε. By the HLS inequality, we have that

$$|\nabla c^\varepsilon_\alpha|_4 \leqslant C_{\mathrm{HLS}} \sum_{\beta\in\mathcal{I}} |b_{\alpha\beta}| \cdot |n^\varepsilon_\beta|_{4/3}.$$

As a result, we have that

$$|n^\varepsilon_\alpha \nabla c^\varepsilon_\alpha|_2 \leqslant |n^\varepsilon_\alpha|_4 \, |\nabla c^\varepsilon_\alpha|_4 \leqslant \sum_\beta C_{\mathrm{HLS}} |b_{\alpha\beta}| \cdot |n^\varepsilon_\alpha|_4 \, |n^\varepsilon_\beta|_{4/3}.$$

Since n^ε_α is bounded independent of ε in the space $L^\infty_t(\delta, T; L^p_x)$, for all $\alpha \in \mathcal{I}$ and all $p \in (1,\infty)$, the product $n^\varepsilon \nabla c^\varepsilon$ is bounded in $L^\infty_t(\delta, T; L^2_x)$. Combining this fact and the estimate (3.14), we have that

$$\sum_\alpha |\nabla n^\varepsilon_\alpha|^2_{L^2_t(\delta,T;L^2_x)}$$

is bounded independent of ε.

Define the space V as

$$H^1 \cap \{f \mid \int f|x|^2 \,\mathrm{d}x < \infty\}.$$

A bounded set in the space V is precompact in L^2. If we now combine the second moment bound (3.5) and the H^1 bound of $(n^\varepsilon_\alpha)_{\alpha\in\mathcal{I}}$, we have that the set $(n^\varepsilon_\alpha)_{\varepsilon\geqslant 0}$, for all $\alpha \in \mathcal{I}$, lies in a compact subspace of L^2 for almost every $t \in [\delta, T]$. Finally, note that the W-estimate where $W := H^{-1}$: $\sum_\alpha |\partial_t n^\varepsilon_\alpha|^2_{L^2_t(\delta,T;H^{-1}_x)}$ is relatively straightforward thanks to the equation (1.1).

Step 3: Proof of the free energy dissipation inequality (1.7). Since the solution to the regularized multi-species PKS system has a decreasing free energy $E[\mathbf{n}^\varepsilon]$, we have that

$$E[\mathbf{n}^\varepsilon(\delta)] \geqslant E[\mathbf{n}^\varepsilon(t)] + \sum_\alpha \int_\delta^t \int n^\varepsilon_\alpha |\nabla \log n^\varepsilon_\alpha - \nabla c^\varepsilon_\alpha|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{3.15}$$

for all $t \in [\delta, T]$. To show (1.7), we need to show proper convergence for each single term in (3.15). We first decompose the free energy dissipation term:

$$\begin{aligned}(3.16)\qquad &\sum_\alpha \int_\delta^T \int_{\mathbb{R}^2} n_\alpha^\varepsilon |\nabla \log n_\alpha^\varepsilon - \nabla c_\alpha^\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = 4\sum_\alpha \int_\delta^T \int_{\mathbb{R}^2} |\nabla \sqrt{n_\alpha^\varepsilon}\,|^2 \,\mathrm{d}x\,\mathrm{d}t + \sum_\alpha \int_\delta^T \int_{\mathbb{R}^2} n_\alpha^\varepsilon |\nabla c_\alpha^\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\quad - 2\sum_{\alpha,\beta} \int_\delta^T \int_{\mathbb{R}^2} b_{\alpha\beta} n_\alpha^\varepsilon n_\beta^\varepsilon \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By the convexity of $f \to \int_{\mathbb{R}^2} |\nabla \sqrt{f}\,|^2 \,\mathrm{d}x$, weak semi-continuity, and the strong convergence of n_α^ε in $L_t^2([\delta, T]; L_x^2)$, we have that the first two terms in (3.16) satisfy the following inequalities:

$$(3.17)\qquad \int_\delta^T \int_{\mathbb{R}^2} |\nabla \sqrt{n_\alpha}\,|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \liminf_{\varepsilon\to 0_+} \int_\delta^T \int_{\mathbb{R}^2} |\nabla \sqrt{n_\alpha^\varepsilon}\,|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

$$(3.18)\qquad \int_\delta^T \int_{\mathbb{R}^2} n_\alpha |\nabla c_\alpha|^2 \,\mathrm{d}x\,\mathrm{d}t = \lim_{\varepsilon\to 0_+} \int_\delta^T \int_{\mathbb{R}^2} n_\alpha^\varepsilon |\nabla c_\alpha^\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Since $(n_\alpha^\varepsilon)_{\varepsilon>0}$ converges strongly in the $L^2([\delta, T] \times \mathbb{R}^2)$ space, the last term on the righthand side of (3.16) converges. Moreover, it can be checked that $S[n_\alpha^\varepsilon(t)] \to S[n_\alpha(t)]$ for almost every $t \in [\delta, T]$. The argument is similar to the one used in [5, Lemma 4.6]. As a result, combining these facts and (3.15), (3.16), (3.17), and (3.18) yields that

$$E[\mathbf{n}(\delta)] \geqslant E[\mathbf{n}(t)] + \sum_\alpha \int_\delta^t \int n_\alpha |\nabla \log n_\alpha - \nabla c_\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s.$$

Now, by the monotone convergence theorem and a Cantor diagonal argument, we have proven (1.7). ◻

Proof of Proposition 3.7. We proceed by contradiction. Assume that at time $T_\star < \infty$, the entropy $\sum_\alpha S[n_\alpha^\varepsilon(T_\star)]$ is uniformly bounded with respect to ε.

First, from (3.3), we directly calculate the time evolution of the entropy:

$$\begin{aligned}(3.19)\qquad \frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha \int n_\alpha^\varepsilon \log n_\alpha^\varepsilon \,\mathrm{d}x &= -\sum_\alpha 4 \int |\nabla \sqrt{n^\varepsilon}\,|^2 \,\mathrm{d}x \\ &\quad - \sum_{\alpha,\beta} b_{\alpha\beta} \int_{n_\alpha^\varepsilon \leqslant K} n_\alpha^\varepsilon \Delta(K^\varepsilon * n_\beta^\varepsilon) \,\mathrm{d}x \\ &\quad - \sum_{\alpha,\beta} b_{\alpha\beta} \int_{n_\alpha^\varepsilon > K} n_\alpha^\varepsilon \Delta(K^\varepsilon * n_\beta^\varepsilon) \,\mathrm{d}x \\ &=: -\sum_\alpha 4 \int |\nabla \sqrt{n_\alpha^\varepsilon}\,|^2 \,\mathrm{d}x + I + II.\end{aligned}$$

The term I in (3.19) can be estimated as follows:

$$I \leqslant \sum_{\alpha,\beta} K|b_{\alpha\beta}|\,|\Delta K^{\varepsilon}|_1 M_{\beta}. \tag{3.20}$$

Recall that $|\Delta K^{\varepsilon}|_1$ is bounded independent of ε, so term I is bounded independent of ε. For the term II in (3.19), we estimate it using Hölder's inequality, Gagliardo-Nirenberg-Sobolev inequality, and Young's inequality as follows:

$$\begin{aligned} II &\leqslant \sum_{\alpha,\beta} |b_{\alpha\beta}|\Big(\int_{n_\alpha^\varepsilon \geqslant K} (n_\alpha^\varepsilon)^2 \,\mathrm{d}x + |\Delta K^{\varepsilon}|_1^2 * |n_\beta^\varepsilon|_2^2\Big) \\ &\leqslant \sum_{\alpha,\beta} |b_{\alpha\beta}|\bigg(\Big(\int_{n_\alpha^\varepsilon \geqslant K} n_\alpha^\varepsilon \,\mathrm{d}x\Big)^{1/2} |n_\alpha^\varepsilon|_3^{3/2} \\ &\qquad + |\Delta K^{\varepsilon}|_1^2\Big(M_\beta K + \int_{n_\beta^\varepsilon \geqslant K} (n_\beta^\varepsilon)^2 \,\mathrm{d}x\Big)\bigg) \\ &\leqslant \sum_{\alpha,\beta} |b_{\alpha\beta}|\bigg(\frac{S_+^{1/2}[n_\alpha]}{(\log K)^{1/2}} C_{\mathrm{GNS}} |n_\alpha^\varepsilon|_1^{1/2} |\nabla\sqrt{n_\alpha^\varepsilon}|_2^2 \\ &\qquad + C_{\mathrm{GNS}} |\Delta K^{\varepsilon}|_1^2 \frac{S_+^{1/2}[n_\beta]}{(\log K)^{1/2}} M_\beta^{1/2} |\nabla\sqrt{n_\beta^\varepsilon}|_2^2 \\ &\qquad + |\Delta K^{\varepsilon}|_1^2 M_\beta K\bigg) \\ &\leqslant \sum_{\alpha,\beta} |b_{\alpha\beta}| C_{\mathrm{GNS}} (1 + |\Delta K^{\varepsilon}|_1^2) \frac{S_+^{1/2}[n_\alpha]}{(\log K)^{1/2}} M_\alpha^{1/2} |\nabla\sqrt{n_\alpha^\varepsilon}|_2^2 \\ &\qquad + \sum_{\alpha,\beta} |b_{\alpha\beta}| \cdot |\Delta K^{\varepsilon}|_1^2 M_\alpha K. \end{aligned} \tag{3.21}$$

Here, S_+ denotes the positive part of the entropy, that is,

$$S_+[f] = \int f \log^+ f \,\mathrm{d}x.$$

Combining the estimates (3.19), (3.20) with (3.21), we end up with

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha S[n_\alpha^\varepsilon] \\ &\quad \leqslant \sum_\alpha \underbrace{\Big(-4 + \sum_\beta |b_{\alpha\beta}| C_{\mathrm{GNS}} (1 + |\Delta K^{\varepsilon}|_1^2) \frac{S_+^{1/2}[n_\alpha^\varepsilon]}{(\log K)^{1/2}} M_\alpha^{1/2}\Big)}_{=:A(t)} |\nabla\sqrt{n_\alpha^\varepsilon}|_2^2 \\ &\qquad + \sum_{\alpha,\beta} |b_{\alpha\beta}| (1 + |\Delta K^{\varepsilon}|_1^2) M_\alpha K. \end{aligned}$$

Since the negative part of the entropy and the second moment are bounded (see (3.6), (3.5)), we have that $A(t)$ can be estimated as follows:

$$A(t) \leqslant -4 + \frac{C_{\mathrm{GNS}}}{(\log K)^{1/2}} \sum_{\beta} |b_{\alpha\beta}|(1 + |\Delta K^{\varepsilon}|_1^2) M_\alpha^{1/2} \Big(S[n_\alpha^\varepsilon(t)] + \frac{1}{2} V(T_\star) \\ + \frac{1}{2}\Big(4 \sum_\alpha M_\alpha + \sum_{\alpha,\beta} \frac{(b_{\alpha\beta})_- M_\alpha M_\beta}{2\pi}\Big)(t - T_\star) + \log(2\pi) M_\alpha + e^{-1}\Big)^{1/2}.$$

Since the entropy $\sum_\alpha S[n_\alpha^\varepsilon]$ is uniformly bounded independent of ε at time $T_\star$, we could take the K large such that $A(t) \leqslant -2$ at time $T_\star$. By continuity, there is a small time τ_ε such that for all $t \in [T_\star, T_\star + \tau_\varepsilon)$,

$$\sum_\alpha S[n_\alpha^\varepsilon(t)] \leqslant \sum_\alpha S[n_\alpha^\varepsilon(T_\star)] + (t - T_\star) \sum_{\alpha,\beta} |b_{\alpha\beta}|(1 + |\Delta K^\varepsilon|_1^2) M_\alpha K,$$

for all $t \in [T_\star, T_\star + \tau_\varepsilon]$. But then we can pick τ independent of ε such that

$$A(t) \leqslant -4 + \frac{C(\mathbf{B}, \mathbf{M})}{(\log K)^{1/2}} \Big(\sum_\alpha S[n_\alpha^\varepsilon(T_\star)] + K\tau + 1 \Big) \leqslant 0.$$

The solution τ to the above inequality is independent of the choice of ε, and $[T_\star, T_\star + \tau) \subset [T_\star, T_\star + \tau_\varepsilon)$ for any ε. Therefore, by Proposition 3.6, we can extend the free energy solution past the $T_\star$, contradicting the maximality of $T_\star$. As a result, we have completed the proof of the proposition. □

4. Smoothness of the Free Energy Solutions

In this section, we prove Theorem 1.8. The proof is similar to the arguments in [13]. For the sake of brevity, we skip some details and emphasize the main differences. The proof is decomposed into several lemmas. We first introduce the concept of Fisher information and renormalized solutions, then prove the L^p integrability of the physically relevant free energy solutions and use standard parabolic equation technique to improve it to C^∞ regularity, and conclude with the proof of the free energy equality.

First, note from the physical restrictions (1.11b) and (1.11c) that we have bounded entropy and free energy dissipation, that is, $\mathcal{A}_t[\mathbf{n}] < \infty$ and bounded second moment $V[\mathbf{n}(t)]$ for all $t \in [0, T_\star)$, where $T_\star$ is the maximal existence time.

Next, we present the following time integral bound for the Fisher information.

Lemma 4.1. *If the conditions in the Theorem 1.8 are satisfied, for any physically relevant free energy solutions to (1.1) and any time $T \in [0, T_\star)$, there exists a constant C_F such that the Fisher information of the solution*

$$F[n_\alpha] := \int_{\mathbb{R}^2} \frac{|\nabla n_\alpha|^2}{n_\alpha} \,\mathrm{d}x$$

is time integrable, that is,

$$\sum_{\alpha\in\mathcal{I}}\int_0^T F[n_\alpha(t)]\,\mathrm{d}t \leqslant C_F\Big(M,T,\mathcal{A}_T[\mathbf{n}],\sup_{t\in[0,T)}\sum_\alpha V_\alpha(t)\Big),\quad T\in[0,T_\star).$$

Proof. The proof is essentially the same as the corresponding one in the single species case. For the sake of brevity, we skip the proof here and refer the interested readers to the proof of Lemma 2.2 and the remark after in the paper [13] for further details. ◻

Remark 4.2. For the supercritical mass case, one can use the relative entropy method to derive the boundness of the entropy and entropy dissipation $\mathcal{A}_T[\mathbf{n}]$ before the blow-up time $T_\star$. We refer the interested reader to the papers [4] and [13] for further details.

The next lemma enables us to take advantage of choosing different renormalizing functions in the later proof.

Lemma 4.3. *Any physically relevant free energy solutions* $\mathbf{n}$ *to* (1.1) *satisfy the following estimate for any times* $0\leqslant t_0\leqslant t_1<T_\star$*:*

$$\begin{aligned}(4.1)\quad &\int_{\mathbb{R}^2}\Gamma(n_\alpha(x,t_1))\,\mathrm{d}x+\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\Gamma''(n_\alpha(x,s))|\nabla n_\alpha(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant\int_{\mathbb{R}^2}\Gamma(n_\alpha(x,t_0))\,\mathrm{d}x+\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\Big((\Gamma'(n_\alpha(x,s))n_\alpha(x,s)-\Gamma(n_\alpha(x,s)))\\ &\qquad\qquad\times\Big(\sum_{\beta\in\mathcal{I}}b_{\alpha\beta}n_\beta(s)\Big)\Big)_+\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant\int_{\mathbb{R}^2}\Gamma(n_\alpha(x,t_0))\,\mathrm{d}x+\sum_{\beta\in\mathcal{I}}|b_{\alpha\beta}|\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\Big|\Gamma'(n_\alpha(x,s))n_\alpha(x,s)\\ &\qquad\qquad-\Gamma(n_\alpha(x,s))\Big|n_\beta(s)\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

where $\Gamma:\mathbb{R}\to\mathbb{R}$ *is an arbitrary convex piecewise* C^1 *function satisfying, with some constant* C_β*, the estimates*

$$(4.2)\quad |\Gamma(u)|\leqslant C_\Gamma(1+u(\log u)_+),\ |\Gamma'(u)u-\Gamma(u)|\leqslant C_\Gamma(1+|u|),\quad\forall\,u\in\mathbb{R}.$$

Remark 4.4. Here, in order to analyse the PKS equation (1.1) with general chemical generation coefficients, we introduce a stronger restriction on the growth of the normalizing function Γ (cf. [13]). Here, we assume that the absolute value of the expression $\Gamma'(u)u-\Gamma(u)$ grows at most linearly at infinity, whereas in the paper [13], it is only assumed that the positive part $(\Gamma'(u)u-\Gamma(u))_+$ grows at most linearly.

Proof. The proof is essentially the same as the proof of Lemma 2.5 in [13]. For the sake of simplicity, we do a formal computation and refer the interested readers to [13] for further justifications. By applying the chain rule, we obtain

$$\begin{aligned}\partial_t\Gamma(n_\alpha) &= \Delta\Gamma(n_\alpha) - \Gamma''(n_\alpha)|\nabla n_\alpha|^2 \\ &\qquad - \nabla c_\alpha \cdot \nabla\Gamma(n_\alpha) - \Gamma'(n_\alpha)\Delta c_\alpha n_\alpha, \quad \forall\, \alpha \in \mathcal{I}.\end{aligned}$$

Now, testing it against an arbitrary smooth function $\chi \in \mathcal{D}(\mathbb{R}^2)$ and using the relation $-\Delta c_\alpha = \sum_\beta b_{\alpha\beta} n_\beta$, we have the following relation:

$$\begin{aligned}&\int_{\mathbb{R}^2}\Gamma(n_\alpha(t_1))\chi\,\mathrm{d}x + \int_{t_0}^{t_1}\int_{\mathbb{R}^2}\Gamma''(n_\alpha)|\nabla n_\alpha(s)|^2\chi\,\mathrm{d}x\,\mathrm{d}s \\ &\quad = \int_{\mathbb{R}^2}\Gamma(n_\alpha(t_0))\chi\,\mathrm{d}x + \int_{t_0}^{t_1}\int_{\mathbb{R}^2}\Big(\Gamma'(n_\alpha)\sum_\beta b_{\alpha\beta}n_\beta n_\alpha \chi \\ &\qquad\quad + \Gamma(n_\alpha)\Delta\chi + \Gamma(n_\alpha)\nabla\cdot(\nabla c_\alpha \chi)\Big)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Rewrite the above relation using integration by parts and the fact that $\Delta c_\alpha = -\sum_\beta b_{\alpha\beta} n_\beta$:

$$\begin{aligned}&\int_{\mathbb{R}^2}\Gamma(n_\alpha(t_1))\chi\,\mathrm{d}x + \int_{t_0}^{t_1}\int_{\mathbb{R}^2}\Gamma''(n_\alpha)|\nabla n_\alpha(s)|^2\chi\,\mathrm{d}x\,\mathrm{d}s \\ &\quad = \int_{\mathbb{R}^2}\Gamma(n_\alpha(t_0))\chi\,\mathrm{d}x + \int_{t_0}^{t_1}\int_{\mathbb{R}^2}\Big([\Gamma'(n_\alpha)n_\alpha - \Gamma(n_\alpha)]\Big(\sum_\beta b_{\alpha\beta}n_\beta\Big)\chi \\ &\qquad\quad + [\Gamma(n_\alpha)\Delta\chi + \Gamma(n_\alpha)\nabla c_\alpha\cdot\nabla\chi]\Big)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Now, taking $\chi \to 1$, we end up with the relation (4.1).

To prove the lemma, one first proves (4.1) with renormalizing functions Γ_i, $i \in \mathbb{N}$, which grow at most linearly at infinity. Next, one can prove the estimate (4.1) with renormalizing functions with super linear growth (4.2) by taking the limit of the inequalities (4.1) subject to approximating linear renormalizing functions $(\Gamma_i)_{i\in\mathbb{N}}$. One uses the Lebesgue dominated convergence theorem to guarantee the convergence of the term

$$\lim_{i\to\infty}\int_{t_0}^{t_1}\Big([\Gamma_i'(n_\alpha)n_\alpha - \Gamma_i(n_\alpha)]\Big(\sum_\beta b_{\alpha\beta}n_\beta\Big)\Big)_+\,\mathrm{d}x\,\mathrm{d}s.$$

However, if the function $\sum_\beta b_{\alpha\beta}n_\beta$ can be either positive or negative, we have to assume that $|\Gamma'(u)u - \Gamma(u)|$ grows at most linearly near infinity. □

Now, we prove the L^p estimate of the solution.

Lemma 4.5. *Consider physically relevant free energy solutions $(n_\alpha)_{\alpha\in\mathcal{I}}$ to equation (1.1) subject to initial data (1.3). Let $t_0 \in [0, T_\star)$ be the time such that*

$\sum_{\alpha\in\mathcal{I}} |n_\alpha(t_0)|_p < \infty$, *for some* $p \in [2,\infty)$. *Then, for all* $t_1 \in [t_0, T] \subset [t_0, T_\star)$, *there exists a constant*

$$C_p := C_p\Big(\mathbf{M}, T, \sum_{\alpha\in\mathcal{I}} |n_\alpha(t_0)|_p, V[\mathbf{n}(t_0)], \mathcal{A}_T\Big)$$

such that

$$\sum_{\alpha\in\mathcal{I}} |n_\alpha(t_1)|_p^p + \frac{p-1}{2p}\sum_{\alpha\in\mathcal{I}}\int_{t_0}^{t_1} |\nabla(n_\alpha^{p/2})|_2^2\,\mathrm{d}s \leqslant C_p, \quad p\in[2,\infty).$$

Proof. The proof is similar to the corresponding one in [13]. We decompose the proof into two steps.

Step 1: We prove a logarithmic improvement to the $L\log L$ integrability. The goal is to show there exists a constant

$$C_{S_2} := C_{S_2}(M, T, \mathcal{A}_T, \sup_{[t_0,T]} V[\mathbf{n}(t)])$$

such that, for any $t_1 \in [t_0, T]$, the estimate

$$\sum_\alpha S_2[n_\alpha(t_1)] \leqslant \sum_\alpha S_2[n_\alpha(t_0)] + C_{S_2}, \quad S_2[f] := \int f(\widetilde{\log} f)^2\,\mathrm{d}x, \tag{4.3}$$

is satisfied, where the $\widetilde{\log}$ function is the logarithmic function truncated from below:

$$\widetilde{\log}\, u := \mathbf{1}_{u\leqslant e} + (\log u)\mathbf{1}_{u>e}.$$

For the sake of notational simplicity, we further introduce the bounded truncated logarithmic function $\widetilde{\log}_K$ as follows:

$$\widetilde{\log}_K(u) := \mathbf{1}_{u\leqslant e} + \mathbf{1}_{e<u\leqslant K}\log u + \mathbf{1}_{u>K}\log K.$$

Since $(\cdot)\widetilde{\log}^2(\cdot)$ does not satisfy the growth constraint (4.2), we approximate it by the function $\Gamma_K(u)$, $K \geqslant e^2$:

$$\Gamma_K(u) := \begin{cases} u(\widetilde{\log}\, u)^2, & u \leqslant K; \\ (2+\log K)u\log u - 2K\log K, & u > K. \end{cases}$$

One can check that the function Γ_K is convex and satisfies the properties (4.2):

$$\Gamma_K''(u) \geqslant 2\frac{\log u}{u}\mathbf{1}_{e\leqslant u\leqslant K} + (2+\log K)\frac{1}{u}\mathbf{1}_{u>K} \geqslant \frac{\widetilde{\log}_K u}{u}\mathbf{1}_{u\geqslant e} \geqslant 0, \tag{4.4}$$

$$|\Gamma_K'(u)u - \Gamma_K(u)| \leqslant 2u\,\widetilde{\log}\,u\mathbf{1}_{u\leqslant K} + 4\log K u\mathbf{1}_{u>K} \leqslant C_K(1+u). \tag{4.5}$$

Now, we estimate the time evolution of $\sum_\alpha \int \Gamma_K(n_\alpha)\,\mathrm{d}x$ by using the renormalization relation (4.1), the positivity of $b_{\alpha\beta}$, (4.4), (4.5), and the definition of $\widetilde{\log}, \widetilde{\log}_K$ as follows:

$$\begin{aligned}
(4.6)\quad & \sum_\alpha \int \Gamma_K(n_\alpha(t_1))\,\mathrm{d}x + \sum_\alpha \int_{t_0}^{t_1}\int \frac{\widetilde{\log}_K(n_\alpha)}{n_\alpha}\mathbf{1}_{n_\alpha\geqslant e}|\nabla(n_\alpha)|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant \sum_\alpha \int \Gamma_K(n_\alpha(t_0))\,\mathrm{d}x \\
&\qquad + \sum_{\alpha,\beta}|b_{\alpha\beta}|\int_{t_0}^{t_1}\int (2n_\alpha \widetilde{\log}\, n_\alpha \mathbf{1}_{n_\alpha\leqslant K} + 4\log K n_\alpha \mathbf{1}_{n_\alpha>K})n_\beta\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant \sum_\alpha \int \Gamma_K(n_\alpha(t_0))\,\mathrm{d}x + 4\sum_{\alpha,\beta}|b_{\alpha\beta}|\int_{t_0}^{t_1}\int n_\alpha \widetilde{\log}_K\, n_\alpha n_\beta\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Now, picking a constant $A\in[e,K]$, we estimate the last term on the righthand side of (4.6) using the GNS inequality as follows:

$$\begin{aligned}
(4.7)\quad & \sum_{\alpha,\beta}|b_{\alpha\beta}|\int n_\alpha \widetilde{\log}_K\, n_\alpha n_\beta\,\mathrm{d}x \\
&= \sum_{\alpha,\beta}|b_{\alpha\beta}|\Big(\int n_\alpha \widetilde{\log}_K\, n_\alpha n_\beta \mathbf{1}_{n_\beta\geqslant A}\,\mathrm{d}x + \int n_\alpha \widetilde{\log}_K\, n_\alpha n_\beta \mathbf{1}_{n_\beta\leqslant A}\,\mathrm{d}x\Big) \\
&\leqslant \sum_{\alpha,\beta}|b_{\alpha\beta}|\Big(\int \frac{(n_\alpha \widetilde{\log}_K\, n_\alpha)(n_\beta \widetilde{\log}_K\, n_\beta)}{\log A}\,\mathrm{d}x + A\int n_\alpha \widetilde{\log}_K\, n_\alpha\,\mathrm{d}x\Big) \\
&\leqslant 2\max_\alpha\Big(\sum_\beta |b_{\alpha\beta}|\Big) \\
&\qquad \times \sum_\alpha\Big(\frac{1}{\log A}\int \Big(\sqrt{n_\alpha \widetilde{\log}_K\, n_\alpha}\Big)^4\,\mathrm{d}x + A(M_\alpha + S_+[n_\alpha])\Big) \\
&\leqslant 2C_{\mathrm{GNS}}^2 \max_\alpha\Big(\sum_\beta |b_{\alpha\beta}|\Big)\times\sum_\alpha\Big(A(M_\alpha+S_+[n_\alpha]) \\
&\qquad + \frac{1}{\log A}\Big(\int n_\alpha \widetilde{\log}_K\, n_\alpha\,\mathrm{d}x\Big)\cdot\Big(\int \Big|\nabla\sqrt{n_\alpha \widetilde{\log}_K\, n_\alpha}\Big|^2\,\mathrm{d}x\Big)\Big) \\
&\leqslant 2C_{\mathrm{GNS}}^2 \max_\alpha\Big(\sum_\beta |b_{\alpha\beta}|\Big) \\
&\qquad \times \sum_\alpha\Big(A(M_\alpha+S_+[n_\alpha]) + \frac{1}{\log A}(M_\alpha+S_+[n_\alpha]) \\
&\qquad \times \Big(\int \frac{|\nabla(n_\alpha)|^2}{n_\alpha}\widetilde{\log}_K\, n_\alpha \mathbf{1}_{n_\alpha\geqslant e}\,\mathrm{d}x + F[n_\alpha]\Big)\Big).
\end{aligned}$$

Now, combining (4.6) and (4.7) and taking K then A large, we have the estimate

$$\begin{aligned}\sum_\alpha \int \Gamma_K(n_\alpha(t_1))\,\mathrm{d}x \leqslant \sum_\alpha \int \Gamma_K(n_\alpha(t_0))\,\mathrm{d}x + 2TC_{\mathrm{GNS}} \max_\alpha \Big(\sum_\beta |b_{\alpha\beta}|\Big) \\ \times \sum_\alpha A(M_\alpha + S_+[n_\alpha]) + 4\sum_\alpha \int_{t_0}^{t_1} F[n_\alpha]\,\mathrm{d}s.\end{aligned}$$

By Lemma 4.3, we have that the estimate (4.3) holds with the constant C_{S_2} depending on T, $\mathcal{A}_T$, and

$$\sup_{0\leqslant t\leqslant T} V[\mathbf{n}(t)].$$

Step 2: As in [13], we define the following renormalization function γ_K, $K \geqslant e$, approximating $(\cdot)^p$:

$$\gamma_K(u) := \begin{cases} \dfrac{u^p}{p}, & u \leqslant K; \\ \dfrac{K^{p-1}}{\log K}(u\log u - u) - \dfrac{p-1}{p}K^p + \dfrac{K^p}{\log K}, & u > K. \end{cases}$$

We can estimate the $|\gamma_K'(u)u - \gamma_K(u)|$ as follows:

$$|\gamma_K'(u)u - \gamma_K(u)| \leqslant \frac{p-1}{p}u^p \mathbf{1}_{u\leqslant K} + 2K^{p-1}u\mathbf{1}_{u>K}.$$

Applying this estimate in the (4.1) yields

$$\begin{aligned}\sum_\alpha \int \gamma_K(n_\alpha(t_1))\,\mathrm{d}x &+ \sum_\alpha \frac{4(p-1)}{p^2}\int_{t_0}^{t_1}\int |\nabla(n_\alpha^{p/2})|^2 \mathbf{1}_{n_\alpha\leqslant K}\,\mathrm{d}x\,\mathrm{d}s \\ &+ \frac{K^{p-1}}{\log K}\sum_\alpha \int_{t_0}^{t_1}\int \frac{|\nabla n_\alpha|^2}{n_\alpha}\mathbf{1}_{n_\alpha\geqslant K}\,\mathrm{d}x\,\mathrm{d}s \\ \leqslant \sum_\alpha \int \gamma_K(n_\alpha(t_0))\,\mathrm{d}x &+ \frac{p-1}{p}\sum_{\alpha,\beta}|b_{\alpha\beta}|\int_{t_0}^{t_1}\int n_\alpha^p \mathbf{1}_{n_\alpha\leqslant K} n_\beta\,\mathrm{d}x\,\mathrm{d}s \\ &+ 2K^{p-1}\sum_{\alpha,\beta}|b_{\alpha\beta}|\int_{t_0}^{t_1}\int n_\alpha \mathbf{1}_{n_\alpha>K} n_\beta\,\mathrm{d}x\,\mathrm{d}s \\ =: \sum_\alpha \int \gamma_K(n_\alpha(t_0))\,\mathrm{d}x &+ T_1 + T_2. \end{aligned} \tag{4.8}$$

For the second term T_1 on the righthand side of (4.8), we decompose it as follows:

$$
\begin{aligned}
T_1 &= \frac{p-1}{p} \sum_{\alpha,\beta} |b_{\alpha\beta}| \int_{t_0}^{t_1} \int n_\alpha^p \mathbf{1}_{n_\alpha \leqslant K} n_\beta (\mathbf{1}_{n_\beta \leqslant K} + \mathbf{1}_{n_\beta > K})\, \mathrm{d}x\, \mathrm{d}s \\
&\leqslant \frac{p-1}{p} \max_\alpha \Big(\sum_\beta |b_{\alpha\beta}| \Big) \sum_\alpha \int_{t_0}^{t_1} \int n_\alpha^{p+1} \mathbf{1}_{n_\alpha \leqslant K}\, \mathrm{d}x\, \mathrm{d}s \\
&\quad + \frac{K^{p-1}(p-1)}{p} \sum_{\alpha,\beta} |b_{\alpha\beta}| \int_{t_0}^{t_1} \int n_\beta^2 \mathbf{1}_{n_\beta > K}\, \mathrm{d}x\, \mathrm{d}s \\
&=: T_{11} + T_{12}.
\end{aligned} \tag{4.9}
$$

The treatment of the T_{11} term is similar to the corresponding one in the proof of Lemma 3.10. It can be estimated by using the Gagliardo-Nirenberg-Sobolev inequality, Chebyshev inequality, and a classical vertical truncation technique with truncation level $A \in (0, K)$ as follows:

$$
\begin{aligned}
T_{11} &= \frac{p-1}{p} \max_\alpha \Big(\sum_\beta |b_{\alpha\beta}| \Big) \\
&\quad \times \sum_\alpha \int_{t_0}^{t_1} \int (\min\{n_\alpha, A\} + (n_\alpha \mathbf{1}_{n_\alpha \leqslant K} - A)_+)^{p+1}\, \mathrm{d}x\, \mathrm{d}s \\
&\leqslant \frac{2^p(p-1)}{p} A^p \max_\alpha \Big(\sum_\beta |b_{\alpha\beta}| \Big) \sum_\alpha M_\alpha (t_1 - t_0) \\
&\quad + \frac{2^p(p-1)}{p} \max_\alpha \Big(\sum_\beta |b_{\alpha\beta}| \Big) \sum_\alpha \iint (n_\alpha \mathbf{1}_{n_\alpha \leqslant K} - A)_+^{p+1}\, \mathrm{d}x\, \mathrm{d}s \\
&\leqslant \frac{2^p(p-1)}{p} A^p \max_\alpha \Big(\sum_\beta |b_{\alpha\beta}| \Big) \sum_\alpha M_\alpha (t_1 - t_0) \\
&\quad + \max_\alpha \Big(\sum_\beta |b_{\alpha\beta}| \Big) \sum_\alpha \frac{C_{\mathrm{GNS}} S_+[n_\alpha]}{\log A} \iint |\nabla (n_\alpha^{p/2})|^2 \mathbf{1}_{n_\alpha \leqslant K}\, \mathrm{d}x\, \mathrm{d}s.
\end{aligned} \tag{4.10}
$$

Here, we can see that if we choose K then A large enough, the second term can be absorbed by the dissipative term on the lefthand side of (4.8). The second term T_{12} in (4.9) has a different flavor. Here, the improved integrability of the solution (4.3) is applied to gain extra smallness on this nonlinear term. As with the paper [13], we apply the bound (4.3), as well as the Sobolev inequality and Cauchy-Schwarz inequality, to estimate the T_{12} term in (4.9) as follows:

$$
\begin{aligned}
T_{12} &\leqslant \frac{4K^{p-1}(p-1)}{p}\sum_{\alpha,\beta}|b_{\alpha\beta}|\int_{t_0}^{t_1}\int (n_\beta - K/2)_+^2 \,\mathrm{d}x\,\mathrm{d}s\\
&\leqslant \frac{4C_S K^{p-1}(p-1)}{p}\sum_{\alpha,\beta}|b_{\alpha\beta}|\int_{t_0}^{t_1}\left(\int |\nabla(n_\beta - K/2)_+|\,\mathrm{d}x\right)^2 \mathrm{d}s\\
&\leqslant \frac{4C_S K^{p-1}(p-1)}{p}\sum_{\alpha,\beta}|b_{\alpha\beta}|\\
&\qquad \times \int_{t_0}^{t_1}\left(\int \frac{|\nabla n_\beta|^2}{n_\beta}(\mathbf{1}_{K\geqslant n_\beta\geqslant K/2} + \mathbf{1}_{n_\beta>K})\,\mathrm{d}x\right)\\
&\qquad \times \left(\int n_\beta \mathbf{1}_{n_\beta\geqslant K/2}\right)\mathrm{d}s\\
&\leqslant \sum_{\alpha,\beta}|b_{\alpha\beta}|\frac{32(p-1)C_S \sup_{t_0\leqslant t\leqslant t_1} S_2[n_\beta(t)]}{p(\log K)^2}\\
&\qquad \times \left(\frac{2^{p+1}}{p^2}\int_{t_0}^{t_1}\int |\nabla(n_\beta^{p/2})|^2 \mathbf{1}_{K/2\leqslant n_\beta\leqslant K}\,\mathrm{d}x\,\mathrm{d}s\right.\\
&\qquad \left. + K^{p-1}\int_{t_0}^{t_1}\int \frac{|\nabla n_\beta|^2}{n_\beta}\mathbf{1}_{n_\beta>K}\,\mathrm{d}x\,\mathrm{d}s\right).
\end{aligned}
$$

Since S_2 is bounded on the time interval $[t_0, t_1]$ (4.3), if K is large enough, these terms can be absorbed by the lefthand side of (4.8).

For the last term T_2 on the righthand side of (4.8), applying the symmetry of the matrix $\mathbf{B}$ (1.5), Hölder's inequality, and Young's inequality, we can estimate it as follows:

$$
\begin{aligned}
T_2 &= 2K^{p-1}\sum_{\alpha,\beta}\int_{t_0}^{t_1}\int n_\alpha \mathbf{1}_{n_\alpha>K}|b_{\alpha\beta}|n_\beta(\mathbf{1}_{n_\beta>K} + \mathbf{1}_{n_\beta\leqslant K})\,\mathrm{d}x\,\mathrm{d}s\\
&\leqslant 4K^{p-1}\max_\alpha\Big(\sum_\beta |b_{\alpha\beta}|\Big)\sum_\alpha\int_{t_0}^{t_1}\int n_\alpha^2 \mathbf{1}_{n_\alpha>K}\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
\tag{4.11}
$$

Now, they are similar to the T_{12} term in (4.9), and we skip the treatment for the sake of brevity.

Combining the estimates (4.9), (4.10), and (4.11), we have from (4.8) that

$$
\begin{aligned}
&\sum_\alpha \int \gamma_K(n_\alpha(t_1))\,\mathrm{d}x + \sum_\alpha \frac{2(p-1)}{p^2}\int_{t_0}^{t_1}\int |\nabla(n_\alpha^{p/2})|^2 \mathbf{1}_{n_\alpha\leqslant K}\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad \leqslant \sum_\alpha \int \gamma_K(n_\alpha(t_0))\,\mathrm{d}x + 2^p A^p \max_\alpha\Big(\sum_\beta |b_{\alpha\beta}|\Big)\sum_\alpha M_\alpha T.
\end{aligned}
$$

Now, we can take A fixed and K to infinity to complete the proof of the lemma. ◻

Next, arguing along the lines of [13], we end up with the conclusion that free energy solutions are classical solutions for all positive time. We quote the following result.

Lemma 4.6 ([13]). *Any physically relevant free energy solutions* $(n_\alpha)_{\alpha\in\mathcal{I}}$ *to* (1.1) *are smooth for any strictly positive time, that is,*

$$n_\alpha \in C^\infty((\delta, T_\star)\times\mathbb{R}^2),$$

for all $\delta > 0$.

We have the following lower semicontinuity of the free energy functional.

Lemma 4.7 ([13]). *Consider any bounded sequences* $(n_{\alpha,k})_{\alpha\in\mathcal{I}}$ *of nonnegative functions in* $L^1_+(\mathbb{R}^2)$ *with finite second moment*

$$\sum_\alpha \int n_{\alpha,k}|x|^2\,\mathrm{d}x < \infty.$$

Assume that $\{n_{\alpha,k}\}_{k=1}^\infty$ *has the same subcritical masses as* n_α*, that is,* $|n_{\alpha,k}|_1 = M_\alpha$*, for all* $\alpha\in\mathcal{I}$*, for all* $k\in\mathbb{N}$*. If there exists a constant* C *such that the free energy* $E[(n_{\alpha,k})_{\alpha\in\mathcal{I}}]$ *is uniformly bounded in* k*, that is,*

$$\sup_k E[(n_{\alpha,k})_{\alpha\in\mathcal{I}}] \leqslant C < \infty,$$

and $\{n_{\alpha,k}\}_{k=1}^\infty$ *converges to* n_α *in* $\mathcal{D}'(\mathbb{R}^2)$ *for all* $\alpha\in\mathcal{I}$*, there hold*

$$\begin{gathered} n_\alpha \in L^1_+(\mathbb{R}^2),\\ \int n_\alpha |x|^2\,\mathrm{d}x < \infty, \qquad \forall\,\alpha\in\mathcal{I},\\ E[(n_\alpha)_{\alpha\in\mathcal{I}}] \leqslant \liminf_{k\to\infty} E[(n_{\alpha,k})_{\alpha\in\mathcal{I}}]. \end{gathered}$$

Equipped with Lemmas 4.6 and 4.7, we turn to the following.

Proof of Theorem 1.8. The smoothness of the solutions was already proven in Lemma 4.6. The proof of the equality in (2.1) is similar to the one in [13]. For the sake of completeness, we detailed the proof as follows.

Since the solution n_α, $\alpha\in\mathcal{I}$, is smooth for all positive time, the following equality holds for all $t_n > 0$, where $t_n \to 0^+$:

$$E[\mathbf{n}(t)] = E[\mathbf{n}(t_n)] + \sum_\alpha \int_{t_n}^t n_\alpha |\nabla \log n_\alpha - \nabla c_\alpha|^2\,\mathrm{d}x\,\mathrm{d}s.$$

Combining this with the Lebesgue dominated convergence theorem, the lower semi-continuity of the functional E proven in the last lemma and the fact that $\mathbf{n}(t_n)$ converges to $\mathbf{n}_0$ weakly in $\mathcal{D}'(\mathbb{R}^2)$, we have that

$$\begin{aligned} E[\mathbf{n}_0] &\leqslant \liminf_{n\to 0} E[\mathbf{n}(t_n)] \\ &\leqslant \lim \left(E[\mathbf{n}(t)] + \sum_\alpha \int_{t_n}^t n_\alpha |\nabla \log n_\alpha - \nabla c_\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s \right) \\ &= E[\mathbf{n}(t)] + \sum_\alpha \int_0^t n_\alpha |\nabla \log n_\alpha - \nabla c_\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Recalling the definition of the free energy solution, the proof of the free energy dissipation equality is completed. □

5. UNIQUENESS OF THE FREE ENERGY SOLUTIONS

After proving the smoothness theorem for the system (1.1), we are ready to prove the uniqueness of the physically relevant free energy solutions $(n_\alpha)_{\alpha\in\mathcal{I}}$. To estimate the deviation between two solutions on a small time interval, some smallness estimates are needed. The following lemma provides the functional space where we could seek for smallness.

Lemma 5.1. *Consider the physically relevant free energy solution* $\mathbf{n}$ *to the system* (1.1)*. The following holds:*

$$\lim_{t\to 0^+} t^{1/4} \sum_\alpha |n_\alpha(t)|_{4/3} = 0. \tag{5.1}$$

Proof. The proof is similar to the one in the paper [13]. Before estimating the norm $t^{1/4}|n_\alpha|_{4/3}$, we collect some estimates which we are going to use. It is enough to consider a short interval $[0,T] \subset [0,T_\star)$. From the assumptions (1.11b), (1.11c) we have that the positive part of the entropy is bounded:

$$\sum_\alpha S_+[n_\alpha(t)] \leqslant C_{L\log L} < \infty, \qquad \forall\, t \in [0,T].$$

Next, we prove the estimate

$$\sum_\alpha |n_\alpha(t)|_2^2 t \leqslant C_{L^2}(\mathbf{B}, \mathbf{M}, |\mathcal{I}|, C_{L\log L}) < \infty, \qquad \forall\, t\in[0,T]. \tag{5.2}$$

The standard L^2 energy estimate yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha |n_\alpha|_2^2 + 2\sum_\alpha |\nabla n_\alpha|_2^2 = \sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta} \int n_\alpha^2 n_\beta \,\mathrm{d}x.$$

Applying the Nash inequality, Gagliardo-Nirenberg-Sobolev inequality, and the vertical truncation technique applied in the proof of Lemma 3.10, we can estimate the righthand side as follows:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha |n_\alpha|_2^2 &\leqslant -\sum_\alpha |\nabla n_\alpha|_2^2 + \sum_{\alpha,\beta} |b_{\alpha\beta}| \cdot |n_\beta|_3^3 \\
&\leqslant -\sum_\alpha |\nabla n_\alpha|_2^2 + \sum_{\alpha,\beta} |b_{\alpha\beta}| \big(|n_\beta \mathbf{1}_{n_\beta \leqslant K}|_3^3 + |n_\beta \mathbf{1}_{n_\beta \geqslant K}|_1^{1/3} |n_\beta \mathbf{1}_{n_\beta \geqslant K}|_4^{8/3} \big) \\
&\leqslant -\sum_\alpha |\nabla n_\alpha|_2^2 \\
&\quad + \sum_{\alpha,\beta} |b_{\alpha\beta}| \left(K^2 M_\beta + \frac{C_{\mathrm{GNS}} \sup_{t\in[0,T]} S_+[\mathbf{n}(t)]^{1/3}}{(\log K)^{1/3}} |n_\beta|_1^{2/3} |\nabla n_\beta|_2^2 \right) \\
&\leqslant -\sum_\alpha \left(1 - \sum_\beta |b_{\alpha\beta}| \frac{C_{\mathrm{GNS}} C_{L\log L}^{1/3}}{(\log K)^{1/3}} M_\alpha^{2/3} \right) |\nabla n_\alpha|_2^2 + \sum_{\alpha,\beta} |b_{\alpha\beta}| K^2 M_\beta \\
&\leqslant -\frac{\left(\sum_\alpha |n_\alpha|_2^2 \right)^2}{2C_N \max_\alpha M_\alpha^2 |\mathcal{I}|} + \sum_{\alpha,\beta} |b_{\alpha\beta}| K^2 M_\beta,
\end{aligned}
$$

where K is a large number chosen such that the coefficient of $|\nabla n_\alpha|_2^2$ is less than $-\frac{1}{2}$. Now, by comparing $|n_\alpha|_2$ with the solution to the super equation

$$
\frac{\mathrm{d}}{\mathrm{d}t} f = -\frac{f^2}{2C_N \max_\alpha M_\alpha^2 |\mathcal{I}|} + K^2 \sum_{\alpha,\beta} |b_{\alpha\beta}| M_\beta, \quad f(0) = \infty,
$$

we obtain (5.2).

Now, we estimate the quantity $t^{1/4} |n_\alpha(t)|_{4/3}$. By Hölder's inequality and the boundedness of the entropy, we have that

$$
\begin{aligned}
(5.3) \qquad (t^{1/4} |n_\alpha|_{4/3})^{4/3} &= t^{1/3} \int n_\alpha^{4/3} \,\mathrm{d}x \\
&\leqslant \left(\int n_\alpha (\log^+ n_\alpha + 2) \,\mathrm{d}x \right)^{2/3} \left(t \int n_\alpha^2 (2 + \log^+ n_\alpha)^{-2} \,\mathrm{d}x \right)^{1/3} \\
&\leqslant C(C_{L\log L}, \mathbf{M}) \left(t \int n_\alpha^2 (2 + \log^+ n_\alpha)^{-2} \,\mathrm{d}x \right)^{1/3}.
\end{aligned}
$$

To estimate the term in the parenthesis, we separate the integral into two parts and use the increasing property of the function $s/(2 + \log^+ s)^2$, the conservation of mass, and (5.2) to estimate each piece:

$$
\begin{aligned}
&t \int n_\alpha^2 (2 + \log^+ n_\alpha)^{-2} \,\mathrm{d}x \\
&\quad \leqslant t \int_{n_\alpha \leqslant R} n_\alpha^2 (2 + \log^+ n_\alpha)^{-2} \,\mathrm{d}x + t \int_{n_\alpha > R} n_\alpha^2 (2 + \log^+ n_\alpha)^{-2} \,\mathrm{d}x \\
&\quad \leqslant t \frac{R}{(2 + \log^+ R)^2} \int_{n_\alpha \leqslant R} n_\alpha \,\mathrm{d}x + \frac{t}{(2 + \log^+ R)^2} \int_{n_\alpha \geqslant R} n_\alpha^2 \,\mathrm{d}x \\
&\quad \leqslant t \frac{MR}{(2 + \log^+ R)^2} + \frac{C_{L^2}}{(2 + \log^+ R)^2}.
\end{aligned}
$$

Now set $R := 1/t$; we then have

$$t\int n_\alpha^2(2+\log^+ n_\alpha)^{-2}\,\mathrm{d}x \leqslant \frac{M+C_{L^2}}{\left(2+\log^+\dfrac{1}{t}\right)^2} \to 0, \quad t\to 0_+.$$

Combining this with (5.3) yields the result. □

Now, we prove Theorem 1.9. Consider the equation (1.1) in the mild form. Since we have smoothness of the free energy solution, we have that the two formulations are equivalent. Suppose that $(n_{\alpha,1})_{\alpha\in\mathcal{I}}, (n_{\alpha,2})_{\alpha\in\mathcal{I}}$ are two solutions subject to the same initial data $n_{\alpha 0}$, $\alpha\in\mathcal{I}$; their difference then satisfies

$$\begin{aligned} &n_{\alpha,2}(t)-n_{\alpha,1}(t)\\ &\quad = -\int_0^t e^{(t-s)\Delta}\nabla\cdot((\nabla c_{\alpha,2}(s)-\nabla c_{\alpha,1}(s))n_{\alpha,2}(s))\,\mathrm{d}s\\ &\quad\quad -\int_0^t e^{(t-s)\Delta}\nabla\cdot(\nabla c_{\alpha,1}(s)(n_{\alpha,2}(s)-n_{\alpha,1}(s)))\,\mathrm{d}s, \quad \forall\,\alpha\in\mathcal{I}. \end{aligned}$$

Define the following quantities:

$$\begin{aligned} Z_{\alpha,\ell}(t) &:= \sup_{0<s\leqslant t} s^{1/4}|n_{\alpha,\ell}(s)|_{4/3}, && \ell=\{1,2\};\\ \Delta_\alpha(t) &:= \sup_{0<s\leqslant t} s^{1/4}|n_{\alpha,2}(s)-n_{\alpha,1}(s)|_{4/3}, && \forall\,\alpha\in\mathcal{I}. \end{aligned}$$

The estimate (5.1) yields that $\lim_{t\to 0_+} Z_{\alpha,\ell}(t)=0$. The $\Delta_\alpha(t)$ can be further decomposed as follows:

$$\begin{aligned} &\Delta_\alpha(T)\\ &\leqslant \sup_{0\leqslant t\leqslant T} t^{1/4}\left|\int_0^t e^{(t-s)\Delta}\nabla\cdot((\nabla c_{\alpha,2}(s)-\nabla c_{\alpha,1}(s))n_{\alpha,2}(s))\,\mathrm{d}s\right|_{4/3}\\ &\quad + \sup_{0\leqslant t\leqslant T} t^{1/4}\left|\int_0^t e^{(t-s)\Delta}\nabla\cdot(\nabla c_{\alpha,1}(s)(n_{\alpha,2}(s)-n_{\alpha,1}(s)))\,\mathrm{d}s\right|_{4/3}\\ &=: \sup_{0\leqslant t\leqslant T} J_{\alpha,1}(t) + \sup_{0\leqslant t\leqslant T} J_{\alpha,2}(t). \end{aligned} \tag{5.4}$$

Now, we estimate the $J_{\alpha,2}$ term in (5.4) using Hölder's inequality, the Hardy-Littlewood-Sobolev inequality, Minkowski's integral inequality, and the heat semigroup estimate as follows:

$$\begin{aligned} J_{\alpha,2}(t) &\leqslant t^{1/4}\int_0^t \frac{C}{(t-s)^{3/4}}|\nabla c_{\alpha,1}|_4\,|n_{\alpha,2}-n_{\alpha,1}|_{4/3}\,\mathrm{d}s\\ &\leqslant \int_0^t C\frac{t^{1/4}}{s^{1/2}(t-s)^{3/4}}\,\mathrm{d}s\sum_{\beta\in\mathcal{I}}|b_{\alpha\beta}|Z_{\beta,1}(t)\Delta_\alpha(t)\\ &\leqslant C\sum_{\beta\in\mathcal{I}}|b_{\alpha\beta}|Z_{\beta,1}(t)\Delta_\alpha(t). \end{aligned} \tag{5.5}$$

Similarly, we can estimate the $J_{\alpha,1}$ term as follows:

$$J_{\alpha,1}(t) \leqslant C \sum_{\beta} |b_{\alpha\beta}| \Delta_\beta(t) Z_{\alpha,2}(t). \tag{5.6}$$

Combining (5.4), (5.6), (5.5), and symmetry of $\mathbf{B}$ (1.5), we have that

$$\begin{aligned}
\sum_{\alpha} \Delta_\alpha(T) &\lesssim \sum_{\alpha,\beta} |b_{\alpha\beta}| \sup_{0\leqslant t\leqslant T} Z_{\beta,1}(t)\Delta_\alpha(t) + \sum_{\alpha,\beta} |b_{\alpha\beta}| \sup_{0\leqslant t\leqslant T} \Delta_\beta(t) Z_{\alpha,2}(t) \\
&\lesssim \sum_{\alpha,\beta} |b_{\alpha\beta}| \sup_{0\leqslant t\leqslant T} \Delta_\alpha(t)(Z_{\beta,1}(t) + Z_{\beta,2}(t)) \\
&\lesssim \max_{\alpha,\beta} |b_{\alpha\beta}| \sum_{\alpha} \Delta_\alpha(T) \Big(\sum_{\beta} \sum_{\ell=1}^{2} Z_{\beta,\ell}(T) \Big).
\end{aligned}$$

Now, since $Z_{\beta,\ell}(t)$ approaches zero as time approaches 0_+ (5.1), there exists a small time T' such that

$$\sum_{\alpha} \Delta_\alpha(T') \leqslant \frac{1}{2} \sum_{\alpha} \Delta_\alpha(T'), \quad T' \in [0, T].$$

Thus, we have $\sum_\alpha \Delta_\alpha \equiv 0$, $\forall\, t \in [0, T']$. The uniqueness now follows if we iterate this argument.

6. Longtime Behavior of the Free Energy Solutions

In this section, we studied the longtime behavior of the multi-species PKS system (1.1). Since the solution becomes instantly smooth, we could assume that the initial data $n_{\alpha 0}$ is $C^\infty \cap L^1$ for all $\alpha \in \mathcal{I}$. We rewrite the equation (1.1) in the self-similar variables

$$X := \frac{x}{R(t)}, \quad \tau := \log R(t), \quad R(t) := \sqrt{1 + 2t}.$$

We define the solutions N_α, C_α in the self-similar variables:

$$n_\alpha(x, t) = \frac{1}{R^2(t)} N_\alpha(X, \tau), \quad c_\alpha(x, t) = C_\alpha(X, \tau).$$

Rewriting the equation (1.1) in the self-similar variables, we obtain that the N_α, C_α satisfy the following equations subject to initial data $N_\alpha(X, \tau = 0)$ $(n_{\alpha 0}(X))$, for all $\alpha \in \mathcal{I}$:

$$\begin{cases} \partial_\tau N_\alpha = \Delta N_\alpha + \nabla \cdot (X N_\alpha) - \nabla \cdot (\nabla C_\alpha N_\alpha), \\ -\Delta C_\alpha = \sum_{\beta \in \mathcal{I}} b_{\alpha\beta} N_\beta. \end{cases} \tag{6.1}$$

To prove Theorem 1.10, we show that the solution N_α to the equation (6.1) is uniformly bounded in time. This is due to the fact that the $L^2(\mathrm{d}x)$ norm of solutions n_α to the original problem and the $L^2(\mathrm{d}X)$ norm of the solutions N_α to the equation (6.1) have the following relation:

$$|n_\alpha|^2_{L^2(\mathrm{d}x)} = \frac{|N_\alpha|^2_{L^2(\mathrm{d}X)}}{R^2(t)} = \frac{|N_\alpha|^2_{L^2(\mathrm{d}X)}}{1+2t}.$$

Therefore, any uniform-in-time bound of $|N_\alpha|_{L^2(\mathrm{d}X)}$ can be translated to decay of $|n_\alpha|_{L^2(\mathrm{d}x)}$. We decompose our proof into several lemmas. First, we show that the second moment of the solutions are uniformly bounded in time.

Lemma 6.1. *Consider the solutions N_α, $\alpha \in \mathcal{I}$ to the equation (6.1). The total second moment is uniformly bounded in time, that is,*

$$\sum_{\alpha\in\mathcal{I}} \int N_\alpha(X,\tau)|X|^2\,\mathrm{d}X \leqslant C_{V,R} < \infty, \qquad \forall\,\tau \in [0,\infty). \tag{6.2}$$

Proof. Similar to the proof of (2.3), we calculate the time evolution of the second moment:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_\alpha \int N_\alpha |X|^2\,\mathrm{d}X \\
&\qquad = -2\sum_\alpha \int N_\alpha |X|^2\,\mathrm{d}X + \Big(\sum_\alpha 4M_\alpha\Big)\left(1 - \frac{Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}]}{8\pi}\right).
\end{aligned}$$

Now we see that the total second moment is bounded:

$$\begin{aligned}
&\sum_\alpha \int N_\alpha |X|^2\,\mathrm{d}X \\
&\qquad \leqslant \max\left\{\frac{1}{2}\Big(\sum_\alpha 4M_\alpha\Big)\left(1 - \frac{Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}]}{8\pi}\right), \sum_\alpha \int (N_\alpha)_0 |X|^2\,\mathrm{d}X\right\}. \qquad \square
\end{aligned}$$

Much as in the proof of the estimate (2.1), we can show that the equation (6.1) has the following decreasing free energy for all $\tau \geqslant 0$:

$$\begin{aligned}
E_R[\mathbf{N}(\tau)] &= \sum_{\alpha\in\mathcal{I}} \int N_\alpha \log N_\alpha\,\mathrm{d}X \\
&\qquad + \sum_{\alpha,\beta\in\mathcal{I}} \frac{b_{\alpha\beta}}{4\pi} \iint \log|X-Y| N_\alpha(X) N_\beta(Y)\,\mathrm{d}X\,\mathrm{d}Y \\
&\qquad + \frac{1}{2}\sum_{\alpha\in\mathcal{I}} \int N_\alpha |X|^2\,\mathrm{d}X \leqslant E_R[\mathbf{N}_0].
\end{aligned}$$

Now, we apply the log-HLS inequality (3.2) to get a bound for the entropy,

$$S_R[\mathbf{N}] = \sum_{\alpha} \int N_\alpha \log N_\alpha \,\mathrm{d}X,$$

obtaining

$$\begin{aligned}
E_R[\mathbf{N}_0] &\geqslant E_R[\mathbf{N}] \\
&\geqslant \sum_{\alpha\in\mathcal{I}} \int N_\alpha \log N_\alpha \,\mathrm{d}X + \sum_{\alpha,\beta\in\mathcal{I}} \frac{(b_{\alpha\beta})_+}{4\pi} \int N_\alpha(X) \log|X-Y| N_\beta(Y)\,\mathrm{d}X\,\mathrm{d}Y \\
&\qquad - \sum_{\alpha,\beta} \frac{(b_{\alpha\beta})_-}{4\pi} \iint_{|X-Y|\geqslant 1} N_\alpha(X) \log|X-Y| N_\beta(Y)\,\mathrm{d}X\,\mathrm{d}Y \\
&\qquad + \frac{1}{2}\int N_\alpha |X|^2 \,\mathrm{d}X \\
&= (1-\theta) \sum_{\alpha\in\mathcal{I}} \int N_\alpha \log N_\alpha \,\mathrm{d}X + \theta\Big(\sum_{\alpha\in\mathcal{I}} \int N_\alpha \log N_\alpha \,\mathrm{d}x \\
&\qquad + \frac{1}{4\pi} \sum_{\alpha,\beta\in\mathcal{I}} \frac{(b_{\alpha\beta})_+}{\theta} \iint N_\alpha(X) \log|X-Y| N_\beta(Y)\,\mathrm{d}X\,\mathrm{d}Y \Big) \\
&\qquad - \sum_{\alpha,\beta} \frac{(b_{\alpha\beta})_-}{4\pi} (M_\alpha V_\beta + M_\beta V_\alpha) + \frac{1}{2}\int N_\alpha |X|^2\,\mathrm{d}X \\
&\geqslant (1-\theta) \sum_{\alpha\in\mathcal{I}} \int N_\alpha \log^+ N_\alpha \,\mathrm{d}X \\
&\qquad - (1-\theta) \int N_\alpha \log^- N_\alpha \,\mathrm{d}X - \theta C_{\mathrm{lHLS}}(\mathbf{B},\mathbf{M}) \\
&\qquad - \sum_{\alpha,\beta} \frac{(b_{\alpha\beta})_-}{4\pi} (M_\alpha V_\beta + M_\beta V_\alpha) + \frac{1}{2}\int N_\alpha |X|^2\,\mathrm{d}X.
\end{aligned}$$

Here, the $\theta \in (0,1)$ is chosen as in the proof of Proposition 3.3. Now, since the second moment is bounded for all time (6.2), we have that $C_{\mathrm{lHLS}} < \infty$ and the negative part of the entropy is uniformly bounded in time, that is,

$$\int N_\alpha(X,\tau) \log^- N_\alpha(X,\tau)\,\mathrm{d}X < C < \infty \quad \forall\, \tau \in [0,\infty),$$

which in turn yields that

$$\begin{aligned}
\sum_{\alpha\in\mathcal{I}} \int N_\alpha(X,\tau) \log^+ N_\alpha(X,\tau)\,\mathrm{d}X \\
< C_{L\log L,R} < \infty, \quad \forall\, \tau \in [0,\infty).
\end{aligned} \tag{6.3}$$

Once the positive part of the entropy is bounded, we estimate the time evolution $\sum_\alpha |(N_\alpha - K)_+|_2^2$ as in the proof of Lemma 3.10:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha |(N_\alpha - K)_+|_2^2 \leqslant \Big(-3+\eta(K)\max_\alpha\Big(\sum_\beta |b_{\alpha\beta}|\Big)C_{\mathrm{GNS}}\Big) \times \sum_\alpha \int |\nabla(N_\alpha - K)_+|^2\,\mathrm{d}X + C(K,\mathbf{B},\mathbf{M})\,|(N_\alpha - K)_+|_2^2 + C(K,\mathbf{B},\mathbf{M}),$$

where $\eta(K) \leqslant C_{L\log L,R}/(\log K)$ is made small enough. Now, we choose the K large enough and apply the Nash inequality to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha |(N_\alpha - K)_+|_2^2 \leqslant -\frac{\Big(\sum_\alpha |(N_\alpha - K)_+|_2^2\Big)^2}{C_N \sum_\alpha |(N_\alpha - K)_+|_1^2\,|\mathcal{I}|} + C(K,\mathbf{B},\mathbf{M})\sum_\alpha |(N_\alpha - K)_+|_2^2 + C(K,\mathbf{B},\mathbf{M}).$$

As $|(N_\alpha - K)_+|_1 \leqslant |N_\alpha|_1 = M_\alpha < \infty$, we have that $\sum_\alpha |(N_\alpha - K)_+|_2 \leqslant C_{L^2,R} < \infty$, for all $\tau \in [0,\infty)$. This completes the proof of Theorem 1.10.

7. MULTI-SPECIES PKS SUBJECT TO NON-SYMMETRICAL COUPLING ARRAYS

7.1. Symmetrizable case. In general, the chemical generation coefficient matrix $\mathbf{B}$ is non-symmetrical. This introduces new challenges in the analysis. We will not cover the general situation in this paper. However, in certain cases, one can symmetrize the system. First, recall the sign function:

$$\mathrm{sign}(f) = \begin{cases} 1, & f > 0, \\ 0, & f = 0, \\ -1, & f < 0. \end{cases}$$

If $\mathrm{sign}(b_{\alpha\beta}) = \mathrm{sign}(b_{\beta\alpha})$ and the matrix $\mathbf{B}$ is three diagonal, that is, $b_{\alpha\beta} \neq 0$ only if $|\alpha - \beta| \leqslant 1$, the system can always be symmetrized. Specifically, all the two species models with $\mathrm{sign}(b_{12}) = \mathrm{sign}(b_{21})$ are symmetrizable. To show the method, we consider system (1.1) subject to a general 3-by-3 matrix

$$\partial_t n_\alpha + \sum_{\beta\in\{1,2,3\}} \nabla\cdot(b_{\alpha\beta}(-\nabla\Delta^{-1})n_\beta n_\alpha) = \Delta n_\alpha, \quad \alpha \in \{1,2,3\},$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \quad \mathrm{sign}(b_{\alpha\beta}) = \mathrm{sign}(b_{\beta\alpha}),\ b_{13} = b_{31} = 0.$$

First, we multiply the equation of n_2 by b_{12}/b_{21} and redefine $\tilde{n}_2 := (b_{12}/b_{21})n_2$ to obtain

$$\begin{aligned}
&\partial_t n_1 + \nabla\cdot(b_{11}(-\nabla\Delta^{-1})n_1 n_1 + b_{21}(-\nabla\Delta^{-1})\tilde{n}_2 n_1) = \Delta n_1;\\
&\partial_t \tilde{n}_2 + \nabla\cdot\Big(b_{21}(-\nabla\Delta^{-1})n_1\tilde{n}_2 +\\
&\qquad\qquad + \frac{b_{21}b_{22}}{b_{12}}(-\nabla\Delta^{-1})\tilde{n}_2\tilde{n}_2 + b_{23}(-\nabla\Delta^{-1})n_3\tilde{n}_2\Big) = \Delta\tilde{n}_2.
\end{aligned}$$

Now, we can perform the same trick on the third equation by multiplying it by $b_{12}b_{23}/(b_{32}b_{21})$ and redefining $\tilde{n}_3 := b_{12}b_{23}n_3/(b_{32}b_{21})$; we thus obtain that

$$\begin{aligned}
&\partial_t \tilde{n}_2 + \nabla\cdot\Big(b_{21}(-\nabla\Delta^{-1})n_1\tilde{n}_2 + \frac{b_{21}b_{22}}{b_{12}}(-\nabla\Delta^{-1})\tilde{n}_2\tilde{n}_2 +\\
&\qquad\qquad + \frac{b_{32}b_{21}}{b_{12}}(-\nabla\Delta^{-1})\tilde{n}_3\tilde{n}_2\Big) = \Delta\tilde{n}_2,\\
&\partial_t \tilde{n}_3 + \nabla\cdot\left(\frac{b_{32}b_{21}}{b_{12}}(-\nabla\Delta^{-1})\tilde{n}_2\tilde{n}_3 + \frac{b_{32}b_{21}b_{33}}{b_{12}b_{23}}(-\nabla\Delta^{-1})\tilde{n}_3\tilde{n}_3\right) = \Delta\tilde{n}_3.
\end{aligned}$$

Now we see that the new coefficient matrix is symmetrical. For the general tridiagonal matrix with $\mathrm{sign}(b_{\alpha\beta}) = \mathrm{sign}(b_{\beta\alpha})$, the symmetrization is similar.

Remark 7.1. This three diagonal chemical generation matrices $\mathbf{B}$ correspond to the fact that there exists a hierarchical structure in the community, in which one species only communicates to their direct neighbors.

7.2. Essentially dissipative case. In this section, we prove Theorem 1.13.

Proof of Theorem 1.13. First, note that if $\mathcal{I}^{(|\mathcal{I}|)} = \mathcal{I}$, then $\mathcal{I}^{(0)}$ is not an empty set. Otherwise, one obtains that $\mathcal{I}^{(|\mathcal{I}|)}$ is an empty set, which is a contradiction. We prove that $\sum_\alpha |n_\alpha(t)|_{L_t^\infty(0,\infty;H_x^s)} \leqslant C_{H^s} < \infty$.

First, we prove the L^∞ bound of the n_α's. We pick all the $\alpha^0 \in \mathcal{I}^{(0)}$, and calculate the time evolution of the $|n_{\alpha^0}|_{2p}^{2p}$, $\forall\, p \in [1,\infty)$, using the fact that $b_{\alpha^0\beta} \leqslant 0$ for all $\beta \in \mathcal{I}$

$$\begin{aligned}
\frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}t}|n_{\alpha^0}|_{2p}^{2p} &= -\frac{2p-1}{p^2}|\nabla(n_{\alpha^0})^p|_2^2 - \frac{2p-1}{2p}\int n_{\alpha^0}^{2p}\Delta c_{\alpha^0}\,\mathrm{d}x\\
&= -\frac{2p-1}{p^2}|\nabla n_{\alpha^0}|_2^2 + \frac{2p-1}{2p}\sum_{\beta\in\mathcal{I}} b_{\alpha^0\beta}\int n_{\alpha^0}^{2p} n_\beta\,\mathrm{d}x \leqslant 0.
\end{aligned}$$

As a result, for any $p \in [1,\infty)$, $|n_{\alpha^0}|_{2p} \leqslant |(n_{\alpha^0})_0|_{2p}$. Since the initial data is in $L^1 \cap L^\infty$, we have that $\max_{\alpha^0\in\mathcal{I}^{(0)}} |n_{\alpha^0}|_{L_t\infty(0,\infty;L_x^\infty)} \leqslant C_{\mathcal{I}^{(0)}} < \infty$. Next, we look at all the α^1 in the set $\mathcal{I}^{(1)}$. Calculating the time evolution of the L^{2p} norm using the Nash inequality, we have that

$$\begin{aligned}\frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}t}\|(n_{\alpha^1})^p\|_2^2 &\leqslant -\frac{2p-1}{p^2}\|\nabla(n_{\alpha^1})^p\|_2^2+\frac{2p-1}{2p}\sum_{\beta\in\mathcal{I}^{(0)}}b_{\alpha^1\beta}\int n_\beta n_{\alpha^1}^{2p}\\ &\leqslant -\frac{2p-1}{p^2}\frac{\|(n_{\alpha^1})^p\|_2^4}{C_N\|(n_{\alpha^1})^p\|_1^2}+\frac{2p-1}{2p}\sum_{\beta\in\mathcal{I}^{(0)}}b_{\alpha^1\beta}|n_\beta|_\infty\,\|(n_{\alpha^1})^p\|_2^2.\end{aligned}$$

Since $|n_\beta|_\infty < C_{\mathcal{I}^{(0)}} < \infty$, $\forall\,\beta\in\mathcal{I}^{(0)}$, we have that

$$\sup_{t\in[0,\infty)}\|n_{\alpha^1}\|_{2p}^{2p}\leqslant\max\Big\{pC_N\sup_{t\in[0,\infty)}\|n_{\alpha^1}\|_p^{2p}\sum_{\beta\in\mathcal{I}^{(0)}}|b_{\alpha^1\beta}|C_{\mathcal{I}^{(0)}},\ \|(n_{\alpha^1})_0\|_{2p}^{2p}\Big\}.$$

Since $|n_{\alpha^1}|_{L^1}=M_{\alpha^1}<\infty$ and $|(n_{\alpha^1})_0|_{L^\infty}<\infty$, by the Moser-Alikakos iteration, we have that $|n_{\alpha^1}|_\infty\leqslant C_{\mathcal{I}^{(1)}}<\infty$. By the same argument, we have that

$$\sup_{t\in[0,\infty)}|n_\alpha(t)|_\infty\leqslant C_\infty<\infty,\qquad\forall\,\alpha\in\mathcal{I}^{(|\mathcal{I}|)}.$$

Since $\mathbf{B}$ is essentially dissipative, $\mathcal{I}^{(|\mathcal{I}|)}=\mathcal{I}$, we have that $|n_\alpha|_{L_t^\infty(0,\infty;L_x^\infty)}\leqslant C_\infty$ for all $\alpha\in\mathcal{I}$.

Next, we estimate the H^s ($2\leqslant s\in\mathbb{N}$) norms of the solutions. Assume we have already obtained the H^{s-1} estimate, that is,

$$|n_\alpha(t)|_{H^{s-1}}\leqslant C_{H^{s-1}}<\infty,\qquad\forall\,t\in[0,\infty).$$

We estimate the time evolution of $\sum_\alpha|\nabla^s n_\alpha|_2^2$ using the GNS inequality and HLS inequality as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha\|\nabla^s n_\alpha\|_2^2 &\leqslant -\sum_\alpha\|\nabla^{s+1}n_\alpha\|_2^2+\sum_\alpha\|\nabla c_\alpha\|_\infty^2\|\nabla^s n_\alpha\|_2^2\\ &\quad+\sum_\alpha\sum_{\ell=2}^{s+1}\|\nabla^\ell c_\alpha\|_4^2\|\nabla^{s+1-\ell}n_\alpha\|_4^2\\ &\lesssim -\sum_\alpha\|\nabla^{s+1}n_\alpha\|_2^2+\sum_{\alpha,\beta}|b_{\alpha\beta}|(M_\beta^2+C_\infty^2)\|\nabla^s n_\alpha\|_2^2\\ &\quad+\sum_{\alpha,\beta}\sum_{\ell=2}^{s+1}|b_{\alpha\beta}|\cdot\|\nabla^{\ell-1}n_\beta\|_{4/3}^2\|\nabla^{s+1-\ell}n_\alpha\|_4^2\\ &\lesssim -\sum_\alpha\frac{\|\nabla^s n_\alpha\|_2^{2+2/s}}{C_{\mathrm{GNS}}\|n_\alpha\|_2^{2/s}}+\sum_\alpha\|\nabla^s n_\alpha\|_2^2+\sum_\alpha\|n_\alpha\|_2^2.\end{aligned}$$

Since $\sum_\alpha |n_\alpha|_{L_t^\infty(0,\infty;L_x^2)} \leqslant C_\infty + \sum_\alpha M_\alpha$, we have that

$$\sum_\alpha |\nabla^s n_\alpha(t)|_2 \leqslant C_{H^s}\Big(C_\infty, \sum_\alpha |\nabla^s n_{\alpha 0}|_2, \mathbf{M}, \mathbf{B}\Big) < \infty$$

for all $t \in [0,\infty)$. This completes the proof of the theorem. □

We conclude with a remark concerning the longtime behavior of the solutions. We can rewrite the equation (1.1) in the self-similar variables as in Section 6 (6.1). Applying similar techniques from the proof of Theorem 1.13 yields that the solutions $\mathbf{n}$ decay in L^2, that is,

$$\sum_\alpha |n_\alpha(t)|_2^2 \leqslant \frac{C}{1+t}, \quad t \in \mathbb{R}_+.$$

Here, C is a constant which only depends on the initial data. We sketch the proof as follows. As in Section 6, the goal is to show that $\sum_\alpha |N_\alpha|^2_{L^2(\mathrm{d}X)}$ is uniformly bounded in time $\tau \in [0,\infty)$. For the sake of simplicity, we use $|\cdot|_p$ to denote $|\cdot|_{L^p(\mathrm{d}X)}$. First, we estimate the L^p norms of the solutions n_{α^0}, $\alpha^0 \in \mathcal{I}^{(0)}$. Combining standard L^p energy estimates, the Nash inequality, and the fact that $b_{\alpha^0\beta} \leqslant 0$ for all $\beta \in \mathcal{I}$ yields that

$$\begin{aligned}
\frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}\tau}|(N_{\alpha^0})^p|_2^2 &- \frac{2p-1}{p^2}|\nabla(N_{\alpha^0})^p|_2^2 \\
&+ \frac{2(2p-1)}{2p}|(N_{\alpha^0})^p|_2^2 + \frac{2p-1}{2p}\sum_\beta b_{\alpha^0\beta}\int N_{\alpha^0}^{2p}N_\beta \,\mathrm{d}X \\
\leqslant -\frac{2p-1}{p^2}&\frac{|(N_{\alpha^0})^p|_2^4}{C_N|(N_{\alpha^0})^p|_1^2} + \frac{2(2p-1)}{2p}|(N_{\alpha^0})^p|_2^2.
\end{aligned}$$

This estimate yields that

$$\sup_{\tau\in[0,\infty)} |N_{\alpha^0}(\tau)|_{2p}^{2p} \leqslant \max\{pC_N \sup_{\tau\in[0,\infty)} |(N_{\alpha^0})(\tau)|_p^{2p}, |N_{\alpha^0}(0)|_{2p}^{2p}\}.$$

Since $|N_{\alpha^0}|_1 = M_{\alpha^0} < \infty$ and $|N_{\alpha^0}(0)|_{L^1\cap L^\infty} < \infty$, we can apply the Moser-Alikakos iteration to obtain that

$$\sup_{\tau\in[0,\infty)} |N_{\alpha^0}(\tau)|_{L^1\cap L^\infty} \leqslant C_{\mathcal{I}^{(0)}} < \infty.$$

Now, applying the same iteration technique as the one in the proof of Theorem 1.13 yields the result.

Remark 7.2. Direct application of the free energy method yields the following general result.

Assume the matrix $\mathbf{B}$ only has positive entries, that is, $\mathbf{B} = \mathbf{B}_+$ case. Define the support of a symmetrical matrix $C_{m\times m}$ to be the indices of the rows such that there are non-zero entries in this row, that is,

$$\mathrm{supp}(C) = \{i \in \{1,2,\dots,m\} \mid C_{ij} \neq 0 \text{ for some } j \in \{1,2,\dots,m\}\}.$$

If there exists a sequence of positive symmetrical matrices $\{\mathbf{B}_\ell\}_{\ell\in\mathcal{L}}$ such that $\sum_{\ell\in\mathcal{L}} \mathbf{B}_\ell = \mathbf{B}$ and

$$Q_{\mathbf{B}_\ell,\mathbf{M}}[\mathcal{J} \cap \mathrm{supp}\,\mathbf{B}_\ell] < Q_{\mathbf{B}_\ell,\mathbf{M}}[\mathcal{I} \cap \mathrm{supp}\,\mathbf{B}_\ell] < C_\ell < 8\pi,$$

for all $\emptyset \neq \mathcal{J} \subsetneq \mathcal{I}$ and $\forall\, \ell \in \mathcal{L}$, and

$$\sum_{\ell\in\mathcal{L}} C_\ell \mathbf{1}_{\alpha\in\mathrm{supp}\,\mathbf{B}_\ell} < 8\pi \quad \forall\, \alpha \in \mathcal{I},$$

then there exists a global solution. A conjecture is that if this condition involving the *strict* inequalities fails—namely, if in some of the strict inequalities the $<$ are replaced by $>$, then there must be a finite time blow-up.

Acknowledgement. This research was supported in part by the National Science Foundation (grant nos. DMS16-13911, RNMS11-07444 (KI-Net)) and by the Office of Naval Research (ONR) (grant nos. N00014-1812465 and N00014-21-12773).

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KEY WORDS AND PHRASES: Patlak-Keller-Segel, chemotaxis, parabolic system.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 35K58, 35K40, 35Q92.

Received: March 6, 2019.